The problem of the 21 students

From puzzle-meister Tom Verhoeff comes the following problem:

> In a class of 21 students, each student is either a liar or a truth-teller. The students know exactly who tells the truth and who doesn't. The teacher asks: "Everybody in this class speaks the truth, right?" .
>
> A student answers: "Unfortunately, there is at least one liar among us." . The teacher looks disappointed and asks for confirmation from the other 20 students. They answer, one by one: "At least 2, sir!" , "At least 3, sir!" , and so on, up to "At least 21, sir!" .
>
> The teacher is shocked! Now, who is actually a liar?

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Roland Backhouse has shown us how to solve these "liar/truth-teller" problems using the predicate calculus. The general strategy is that if person P makes a statement S then we have the following equivalence:

$$\text{“ P is a truth-teller ”} \quad \equiv \quad S \qquad .$$

In the given problem we do not have a single speaker, but 21 . Therefore we number the students i $(0 \le i < 21)$, in the order of their speaking, and introduce a sequence of booleans $T.i$ $(0 \le i < 21)$ defined as follows:

$$T.i \quad \equiv \quad \text{“ student } i \text{ is a truth-teller ”} \qquad .$$

Next, we formalize the students' statements:

student i 's statement
$\equiv$ { see above (note that we started numbering from 0) }
" there are at least $i+1$ liars among us "
$\equiv$ { introducing $\mathcal{L}_{\text{student}}$ for the number of liars among the students }
$i+1 \ \le \ \mathcal{L}_{\text{student}}$
$\equiv$ { arithmetic }
$i < \mathcal{L}_{\text{student}}$.

Thus, using Backhouse's technique sketched above, we have:

(0) $T.i \quad \equiv \quad i < \mathcal{L}_{\text{student}}$.

With this formalization, our goal is to determine each value of the sequence T .

A helpful heuristic: When a sequence of booleans enters the picture, it is a good idea to investigate its simple monotonicity properties. Since transitive relations have well-known monotonicity properties, it is an especially good idea when that sequence is given in terms of a transitive relation.

Let us apply this heuristic to the sequence T . Since $i < \mathcal{L}_{\text{student}}$ is shrinking as i increases, $T.i$ is strengthening as i increases. Therefore, T is a sequence of '**true**'s followed by a sequence of '**false**'s . So we can achieve our goal by calculating the number of '**true**'s or the number of '**false**'s in T .

The number of '**false**'s equals $\mathcal{L}_{\text{student}}$, hence the number of '**true**'s equals $21 - \mathcal{L}_{\text{student}}$. We calculate with the latter because we can express it simply using T :

$$
\begin{array}{ll}
 & 21 - \mathcal{L}_{\text{student}} \\
= & \{ \text{ the number of `}\textbf{true}\text{'s in } T \ \} \\
 & \langle \# i : \ 0 \le i \ \wedge \ i < 21 \ : \ T.i \ \rangle \\
= & \{ \ (0) \ \} \\
 & \langle \# i : \ 0 \le i \ \wedge \ i < 21 \ : \ i < \mathcal{L}_{\text{student}} \ \rangle \\
= & \{ \text{ transitivity, using } \mathcal{L}_{\text{student}} \le 21 \ \} \\
 & \langle \# i : \ 0 \le i \ : \ i < \mathcal{L}_{\text{student}} \ \rangle \\
= & \{ \text{ property of } \# \text{ , using } 0 \le \mathcal{L}_{\text{student}} \ \} \\
 & \mathcal{L}_{\text{student}} \qquad .
\end{array}
$$

By algebra this is equivalent to $2 * \mathcal{L}_{\text{student}} \ = \ 21$, which equivales **false** for integer $\mathcal{L}_{\text{student}}$. So our givens imply **false** , and thus the problem as formalized has no consistent solution.

Observe that this conclusion only depends upon the fact that 21 is odd; the conclusion would be equally grim for 555 students. And if the number of students is even, we find that the first half of the students are truth-tellers, and the latter half are liars. For example, with the number of students equal to 20 , the above calculation yields $2 * \mathcal{L}_{\text{student}} \ = \ 20$, which equivales $\mathcal{L}_{\text{student}} \ = \ 10$.

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Our options now are either to be satisfied with the above and consider the question answered, or to investigate a different formalization. Originally we chose the latter option, and since it worked out nicely, we pursue that option here as well.

In our previous investigation, we took "among us" in the problem statement to mean "among the students" . In this investigation we take "among us" to mean "among the class, teacher included" . Introducing $\mathcal{L}_{\text{total}}$ for the number of liars among the entire class, teacher included, we rewrite (0) as (1) :

$$(1) \quad T.i \quad \equiv \quad i < \mathcal{L}_{\text{total}} \qquad .$$

This is the only change to our formalization. Ostensibly, our goal now is to determine each value of the sequence T , and the status of the teacher.

Since a sub-goal of this goal is the same as our previous goal, we would like to investigate this sub-goal along the lines of the previous investigation. Towards that end, we observe that T is still a strengthening sequence, hence has the same topology, and hence we investigate this sub-goal, as before, by calculating with the number of '**true**'s in T . We calculate:

$$
\begin{array}{ll}
 & 21 - \mathcal{L}_{\text{student}} \\
= & \{ \text{ the number of } \text{'}\textbf{true}\text{'s in } T \ \} \\
 & \langle \# i :\ 0 \le i \ \wedge \ i < 21 \ :\ T.i \,\rangle \\
= & \{ \ (1) \ \} \\
 & \langle \# i :\ 0 \le i \ \wedge \ i < 21 \ :\ i < \mathcal{L}_{\text{total}} \,\rangle \\
= & \{ \ \bullet\, \mathcal{L}_{\text{total}} \ \le \ 21 \ \ , \text{ see below } \} \\
 & \langle \# i :\ 0 \le i \ :\ i < \mathcal{L}_{\text{total}} \,\rangle \\
= & \{ \text{ property of } \# \text{ , using } 0 \le \mathcal{L}_{\text{total}} \ \} \\
 & \mathcal{L}_{\text{total}} \\
= & \{ \text{ introducing } \mathcal{L}_{\text{teacher}} \text{ for the number of liars ``among the teacher''} \ \} \\
 & \mathcal{L}_{\text{student}} \ + \ \mathcal{L}_{\text{teacher}} \qquad .
\end{array}
$$

By algebra this is equivalent to:

$$2 * \mathcal{L}_{\text{student}} + \mathcal{L}_{\text{teacher}} = 21 \quad ,$$

which by the types of the variables implies:

$$\mathcal{L}_{\text{student}} = 10 \quad \text{and} \quad \mathcal{L}_{\text{teacher}} = 1 \quad .$$

In other words, the first 11 students are truth-tellers, and the last 10 students and the teacher are liars. As before, a similar pattern holds for any odd number of students. With an even number of students, we find that the first half are truth-tellers, the latter half are liars, and the teacher is not a liar. (We cannot conclude, however, that the teacher is a truth-teller, because according to the problem statement, only the students are either truth-tellers or liars.)

To finish, we show that we have $0 \leq \mathcal{L}_{\text{total}} \leq 21$, which is used in the calculation above. The type of $\mathcal{L}$ implies $0 \leq \mathcal{L}_{\text{total}} \leq 22$, so we establish $0 \leq \mathcal{L}_{\text{total}} \leq 21$ by showing that $\mathcal{L}_{\text{total}} \neq 22$. This, in turn, we can establish by showing $T.i$ for some i . By the topology of T there is an obvious choice for i , namely $i = 0$, since $T.i \Rightarrow T.0$ for all i . Hence we calculate:

$T.0$

$\equiv$ { (0) }

$0 < \mathcal{L}_{\text{total}}$

$\equiv$ { $\mathcal{L}_{\text{total}}$ equals the number of liars }

"there exists at least 1 liar"

$\Leftarrow$ { instantiation }

"student 0 is a liar"

$\equiv$ { definition of T , students are either truth-tellers or liars }

$\neg T.0$.

By predicate calculus, $\neg T.0 \Rightarrow T.0$ equivales $T.0$, and so our investigation is complete.

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Our options now are either to be satisfied with the above and consider the question answered, or investigate a different formalization. Originally, we chose the former option because it was such a nice day and there was frisbee to be played. Since it worked out nicely we pursue that option here as well.

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