An exercise in calculation

Last week, Wim Feijen provided Jeremy Weissmann and me with the following defining properties of functions $\uparrow$ and **max** . We have for any a , b , and m of the appropriate types:

(0) $\quad a \uparrow b \ \leq \ m \quad \equiv \quad a \leq m \ \wedge \ b \leq m$

(1) $\quad a \,\mathbf{max}\, b \ = \ m \quad \equiv \quad (a = m \ \vee \ b = m) \quad \wedge \quad a \leq m \quad \wedge \quad b \leq m$

We are required to prove :

(2) $\quad \mathbf{max} = \uparrow$

To this end, we calculate for any a , b :

$a \,\mathbf{max}\, b \quad = \quad a \uparrow b$

$\equiv \quad \{ \ (1) \ , \text{with } \ m := \ a \uparrow b \ \}$

$(a = a \uparrow b \quad \vee \quad b = a \uparrow b) \quad \wedge \quad a \ \leq \ a \uparrow b \quad \wedge \quad b \ \leq \ a \uparrow b$

We see that the last two conjuncts match the shape of (0) quite nicely. This invites us to investigate :

$a \ \leq \ a \uparrow b \quad \wedge \quad b \ \leq \ a \uparrow b$

$\equiv \quad \{ \ (0) \ , \text{with } \ m := \ a \uparrow b \ \}$

$a \uparrow b \ \leq \ a \uparrow b$

$\equiv \quad \{ \ \leq \ \text{is reflexive} \ \}$

true

Thus, we have also proved for any a , b

(3) $\quad a \ \leq \ a \uparrow b$

and

(4) $\quad b \ \leq \ a \uparrow b \quad .$

Our remaining proof obligation is to transform the first conjunct

$$a = a \uparrow b \quad \vee \quad b = a \uparrow b$$

to **true** in a series of non-weakening steps.

We are not given any relation between $\uparrow$ and $=$. However, (0) relates $\uparrow$ to $\leq$ and $\wedge$. Thus we try to transform $=$ to $\leq$ in a non-weakening step. Indeed, we can do so for an anti-symmetric $\leq$. We calculate for only one of the disjuncts, the result for the other follows by symmetry.

$a = a \uparrow b$

$\Leftarrow$ { for an anti-symmetric $\leq$ }

$a \ \leq \ a \uparrow b \quad \wedge \quad a \uparrow b \ \leq \ a$

$\equiv$ { (3) on the first conjunct, identity of $\wedge$; (0) on the second with $m := a$ }

$a \ \leq \ a \quad \wedge \quad b \ \leq \ a$

$\equiv$ { $\leq$ is reflexive }

$b \leq a$

Due to the symmetry of the two disjuncts, we may conclude

$a = a \uparrow b \quad \vee \quad b = a \uparrow b$

$\Leftarrow$ { by the above calculation and symmetry }

$b \leq a \quad \vee \quad a \leq b$

$\equiv$ { for a linear $\leq$ }

true

This completes our proof for a reflexive, anti-symmetric, and linear $\leq$. I think that this calculation is a wonderful example of the kind of syntactic manipulation a good calculationist should be able to perform. The calculation was carried out in four phases, so to speak. Recalling that we wished to transform the demonstrandum to **true** in a series of non-weakening steps, we immediately realize that we need to eliminate **max** and $\uparrow$ from the picture. From the get-go we used (1) to eliminate **max** . In the second phase, we succeeded in eliminating two instances of $\uparrow$. In the third phase we had two more instances of $\uparrow$ to eliminate and we massaged the formulae into a shape where we made this possible. Finally, we assumed a property of $\leq$ to establish the result.

This sort of streamlined reasoning becomes almost second nature to a seasoned calculator. I myself did not attack the problem in the above fashion initially. I just did not parse the givens in a way that enabled me to make the first manipulation. Once Jeremy Weissmann pointed out that I was too constrained in my parsing of the givens, the rest of the calculation followed.

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There is a another theorem related to **max** whose proof serves to re-enforce the lesson learnt above. Given (1) we are required to prove for any a and b of the appropriate type :

(5) $a \leq b \quad \vee \quad b \leq a$

We note that the disjuncts of our demonstrandum bear similarity with $a \leq m$ and $b \leq m$ in (1) . Thus we investigate, focussing on $a \leq b$ for the moment

true

$\Rightarrow$ { (1) with $m := b$ }

$a \,\mathbf{max}\, b \;=\; b \quad\equiv\quad (a = b \;\vee\; b = b) \quad\wedge\quad a \;\leq\; b \quad\wedge\quad b \;\leq\; b$

$\equiv$ { $=$ is reflexive, zero of $\vee$; $\leq$ is reflexive ; identity of $\wedge$ }

$a \,\mathbf{max}\, b \;=\; b \quad\equiv\quad a \;\leq\; b$

Thus we have

(6) $a \,\mathbf{max}\, b \;=\; b \quad\equiv\quad a \;\leq\; b$

and, by symmetry,

(7) $a \,\mathbf{max}\, b \;=\; a \quad\equiv\quad b \;\leq\; a$

Summarizing, we have

$a \leq b \quad \vee \quad b \leq a$

$\equiv$ { (6) and (7) }

$a \,\mathbf{max}\, b \;=\; b \quad\vee\quad a \,\mathbf{max}\, b \;=\; a$

Again, we note that the above has syntactic similarity with the disjuncts in (1) with $m := a \,\mathbf{max}\, b$. Since (1) is our only given we are all but forced to create a context where it is applicable. Remembering that we are in a strengthening chain, we calculate :

$$a \,\mathbf{max}\, b \;=\; b \quad\vee\quad a \,\mathbf{max}\, b \;=\; a$$

$\Leftarrow$ { aiming to make another appeal to (1) }

$$(a \,\mathbf{max}\, b \;=\; b \quad\vee\quad a \,\mathbf{max}\, b \;=\; a) \quad\wedge\quad a \;\leq\; a \,\mathbf{max}\, b \quad\wedge\quad b \;\leq\; a \,\mathbf{max}\, b$$

$\equiv$ { (1) with $m := a \,\mathbf{max}\, b$ }

$$a \,\mathbf{max}\, b \;=\; a \,\mathbf{max}\, b$$

$\equiv$ { $=$ is reflexive }

true

And we are done!

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I think the two calculations presented here are exemplary of the kind of manipulations an effective calculator must be able to perform. They served as good exercises in sharpening my formula parsing skills and it is for this reason that I think they are worth recording.
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