Punctual predicate transformers are as conjunctive as they are monotonic

Recently, Jeremy Weissmann mentioned in an email that for predicate transformer f

$$\begin{array}{ll} & f \text{ is punctual} \\ (0) & \Rightarrow \\ & (f \text{ is conjunctive} \;\equiv\; f \text{ is monotonic}) \,. \end{array}$$

He encountered it in a paper by Lex Bijlsma and Rob Nederpelt [BN98], where it is stated that a proof of this theorem "using only algebraic concepts, requires more than two pages of calculations". The purpose of this note is to refute that claim, and, moreover, to show that, from the shape of the demonstranda, the proof is self-conducting.

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We first define the key notions of the theorem.

- f is a predicate transformer — a __function__ therefore —:

  $\equiv$

  $\langle \forall x,y :: [x \equiv y] \Rightarrow [f.x \equiv f.y] \rangle$.

- f is __punctual__

  $\equiv$

  $\langle \forall x,y :: [\,(x \equiv y) \Rightarrow (f.x \equiv f.y)\,] \rangle$.

- f is <u>conjunctive</u>
$\equiv$
$\langle \forall x,y :: [\, f.(x \wedge y) \equiv f.x \wedge f.y \,] \rangle$.

- f is <u>monotonic</u>
$\equiv$
$\langle \forall x,y :: [\, x \Rightarrow y \,] \Rightarrow [\, f.x \Rightarrow f.y \,] \rangle$.

* * *

Half of theorem (0)'s consequent has nothing to do with punctuality, viz.

f is conjunctive $\Rightarrow$ f is monotonic ,

and its proof is straightforward and, by now, absolutely standard:

for conjunctive f, we have
for any x, y

$$
\begin{array}{ll}
 & [\, f.x \Rightarrow f.y \,] \\
\equiv & \quad \{ \Rightarrow / \wedge \} \\
 & [\, f.x \wedge f.y \equiv f.x \,] \\
\equiv & \quad \{ f \text{ is conjunctive} \} \\
 & [\, f.(x \wedge y) \equiv f.x \,] \\
\Leftarrow & \quad \{ \text{Leibniz, viz. } f \text{ is a function} \} \\
 & [\, x \wedge y \equiv x \,] \\
\equiv & \quad \{ \Rightarrow / \wedge \} \\
 & [\, x \Rightarrow y \,]
\end{array}
$$

Thus, (0) reduces to

$$\begin{array}{l} f \text{ is punctual} \\ \Rightarrow \\ (f \text{ is conjunctive } \Leftarrow f \text{ is monotonic}), \end{array}$$

or -equivalently- to

$$(1)\qquad \begin{array}{l} f \text{ is punctual} \wedge f \text{ is monotonic} \\ \Rightarrow \\ f \text{ is conjunctive}, \end{array}$$

and the proof of (1) is as follows:

for punctual and monotonic f, we have for any x, y

$$\begin{array}{lll}
 & & [\, f.(x \wedge y) \equiv f.x \wedge f.y \,] \\
(i) & \equiv & \{ f \text{ is monotonic, hence } [\, f.(x \wedge y) \Rightarrow f.x \wedge f.y \,] \} \\
 & & [\, f.x \wedge f.y \Rightarrow f.(x \wedge y) \,] \\
(ii) & \equiv & \{ \text{pred. calc.} \} \\
 & & [\, (f.x \Rightarrow f.(x \wedge y)) \vee (f.y \Rightarrow f.(x \wedge y)) \,] \\
(iii) & \equiv & \{ f \text{ is monotonic, hence } [\, f.x \Leftarrow f.(x \wedge y) \,] \text{ and } [\, f.y \Leftarrow f.(x \wedge y) \,] \} \\
 & & [\, (f.x \equiv f.(x \wedge y)) \vee (f.y \equiv f.(x \wedge y)) \,] \\
 & \Leftarrow & \{ f \text{ is punctual} \} \\
 & & [\, (x \equiv x \wedge y) \vee (y \equiv x \wedge y) \,] \\
 & \equiv & \{ \text{pred. calc.} \}
\end{array}$$

$$
\begin{array}{ll}
\equiv & \{ \Rightarrow/\wedge \text{ twice} \} \\
& [\ (x \Rightarrow y) \vee (y \Rightarrow x)\ ] \\
\equiv & \{ \text{pred. calc.} \} \\
& \text{true} \ .
\end{array}
$$

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As said, the proof is self-conducting. In step (i) punctuality is not applicable due to the multiple occurrence of f in the right-hand side of the first line. But monotonicity is, and applying it is advantageous since it formally weakens the demonstrandum. Step (ii) separates the two f's, which is necessary for a later application of punctuality. Step (iii) uses monotonicity once more in order to prepare for application of f's punctuality.

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[BN98] Lex Bijlsma and Rob Nederpelt, Dijkstra-Scholten predicate calculus: concepts and misconceptions, Acta Informatica 35, 1007-1036 (1998)