A nice little problem communicated by J. Misra

In a recent e-mail, Jayadev Misra formulated the following problem:

"A biequivalence relation is an equivalence relation which has at most two equivalence classes. Show that $\sim$ is a biequivalence relation iff

(0) $\langle\forall x,y,z:: \ x\sim y \ \equiv \ y\sim z \ \equiv \ z\sim x\rangle$.",

and he challenged us to construct a nice calculational proof. This note describes one, as concocted at the beginning of this afternoon's ETAC-session.

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A more traditional way to capture that an equivalence relation $\sim$ has at most two equivalence classes is by stating the validity of

(1) $\langle\forall x,y,z:: \ x\sim y \ \vee \ y\sim z \ \vee \ z\sim x\rangle$.

And what we shall do is to show the equivalence of (0) and (1) by showing the equivalence of the respective terms.

In order to investigate how far we can get without using specificities of $\sim$, we name the various constituent subexpressions of (0) and (1):

P: $x \sim y$
Q: $y \sim z$
R: $z \sim x$,

and thus we are heading for a proof of

(2) $[P \equiv Q \equiv R \equiv P \vee Q \vee R]$.

This is definitely not a theorem of the predicate calculus — e.g. take $P \equiv true$ — and therefore knowledge about $\sim$ will have to creep in sooner or later. This forms an incentive to prove (2) by mutual implication, because the two implications may depend on different facts about $\sim$. Therefore we disentangle (2) into (2a) and (2b):

(2a) $[(P \equiv Q \equiv R) \Rightarrow P \vee Q \vee R]$

(2b) $[(P \equiv Q \equiv R) \Leftarrow P \vee Q \vee R]$.

Re (2a)

$[\ (P \equiv Q \equiv R) \Rightarrow P \vee Q \vee R\ ]$
$\equiv$ { shunting, heading for $\wedge$ over $\equiv\equiv$ }
$[\ (P \equiv Q \equiv R) \wedge (\neg P \wedge \neg Q \wedge \neg R) \Rightarrow false\ ]$
$\equiv$ { $\wedge$ over $\equiv\equiv$, and using $X \wedge \neg X \equiv false$ }
$[(false \equiv false \equiv false) \Rightarrow false]$
$\equiv$ { pred. calc. }
$true$

$\boxtimes$

So, (2a) does not depend on $\sim$ at all !

Re (2b)

By symmetry, it suffices to show

$$[(P \equiv Q \equiv R) \Leftarrow P] ,$$

for which we proceed as follows:

$$
\begin{array}{ll}
 & [\, P \Rightarrow (P \equiv Q \equiv R)\,] \\
\equiv & \quad \{\ (P\Rightarrow) \text{ over } \equiv \ \} \\
 & [\, P \Rightarrow P \;\equiv\; P \Rightarrow Q \;\equiv\; P \Rightarrow R \,] \\
\equiv & \quad \{\ \text{unit of } \equiv \ \} \\
 & [\, P \Rightarrow Q \;\equiv\; P \Rightarrow R \,] \\
\equiv & \quad \{\ (P\Rightarrow) \text{ over } \equiv \ \} \\
 & [\, P \Rightarrow (Q \equiv R)\,] \\
\equiv & \quad \{\ \text{definitions of } P, Q, \text{ and } R \ \} \\
 & [\, x \sim y \;\Rightarrow\; (y \sim z \equiv z \sim x)\,] \\
\equiv & \quad \{\ \text{by symmetry and transitivity of } \sim \ \} \\
 & \text{true}
\end{array}
$$

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Thanks to Jayadev Misra for providing this nice example, and to Rik van Geldrop for suggesting to exploit the symmetry in showing (2b).

W.H.J. Feijen

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