An experiment in solving a fixed-point equation

Preliminaries

We deem the following concepts and theorems to be known and established:

- q is the strongest solution of equation $x : C.x$ means

  $\langle \forall x : C.x : [q \Rightarrow x] \rangle$

  — the extremity of q —,

  and

  $C.q$ — q solves — ;

- the theorem of Knaster and Tarski:

  for monotonic f, the equations $x : [f.x \Rightarrow x]$ and $x : [f.x \equiv x]$ have the same (unique) strongest solution;

- the composition operator, denoted $;$ ("semi"), is a binary operator that is associative, universally disjunctive in both arguments, and that has J as its two-sided unit element;

- $x: [t \vee x;s \Rightarrow x]$ has $t;*s$ as its strongest solution; (0)

  $x: [t \vee s;x \Rightarrow x]$ has $*s;t$ as its strongest solution; (1)

- $[*s;s \Rightarrow *s]$ (2a)

  $[t \Rightarrow t;*s]$ (2b)

  $[*s;*s \equiv *s]$. (2c)

The experiment

We consider equation

$$x: [t \vee x;s;x \Rightarrow x] .$$

Since the antecedent, considered as a function of x, is monotonic, the equation has a strongest solution, q say. The question is whether we can express q as a regular expression.

Our starting point is the definition of q – what else! – viz.

$$\langle \forall x : [t \vee x;s;x \Rightarrow x] : [q \Rightarrow x] \rangle$$

– the extremity of q –,

and

$$[t \vee q;s;q \Rightarrow q]$$

– the solves-part of q – .

We might start focussing on the solves-part of q, trying to transform it into a shape

$$[\text{some regular expression in } s \text{ and } t \Rightarrow q],$$

but on closer scrutiny this looks quite cumbersome because it requires us to get rid of the two q's in the antecedent. Of course this can be done along the lines of AvG183/WF270 ("Exploiting universal junctivity"), using the composition's universal disjunctivity. But it is not clear then, how regular expressions can enter the picture.

Therefore, let us focus on the extremity of q, and head for a calculation of the form

$$[t \vee x;s;x \Rightarrow x]$$
$$\Rightarrow$$
$$[\text{some regular expression in } s \text{ and } t \Rightarrow x].$$

In building this weakening chain, we must however be quite cautious in performing weakening steps, lest we may end up with a regular expression that is too strong for satisfying the solves-part of q.

$$
\begin{array}{ll}
 & [t \vee x;s;x \Rightarrow x] \\
\equiv & \quad \{\text{predicate calculus}\} \\
 & [t \Rightarrow x] \wedge [x;s;x \Rightarrow x] \\
\Rightarrow & \quad \{\text{transitivity of } \Rightarrow, \qquad\qquad (*) \\
 & \quad\ \text{monotonicities (of ; in particular)}\} \\
 & [t \Rightarrow x] \wedge [x;s;t \Rightarrow x] \\
\equiv & \quad \{\text{predicate calculus}\} \\
 & [t \vee x;s;t \Rightarrow x] \\
\Rightarrow & \quad \{\text{extremity of (0) with } s:=s;t\} \\
 & [t;*(s;t) \Rightarrow x]
\end{array}
$$

Now our choice for q will be

$$[q \equiv t;*(s;t)] ,$$

and next we investigate whether it satisfies the solves-part.

<u>Remark</u> If in the step marked (*), we had replaced the other x with t, we would have ended with $[*(t;s);t \Rightarrow x]$, thus by an appeal to the extremity of (1).
<u>End</u> of Remark.

$$
\begin{array}{ll}
 & [t \vee q;s;q \Rightarrow q] \\
\equiv & \quad \{\text{predicate calculus}\} \\
 & [t \Rightarrow q] \wedge [q;s;q \Rightarrow q] \\
\equiv & \quad \{\text{our choice for } q\} \\
 & [t \Rightarrow t;*(s;t)] \\
 & \quad \wedge\ [t;*(s;t);s;t;*(s;t) \Rightarrow t;*(s;t)] \\
\equiv & \quad \{\text{first conjunct is true, by (2b)} \\
 & \quad\ \text{with } s:=s;t\}
\end{array}
$$

$$[\,t\,;\,*(s;t)\,;\,s\,;\,t\,;\,*(s;t) \Rightarrow t\,;\,*(s;t)\,]$$

$\Leftarrow$ $\{$ (2a) with $s := s;t$ $\}$

$$[\,t\,;\,*(s;t)\,;\,*(s;t) \Rightarrow t\,;\,*(s;t)\,]$$

$\equiv$ $\{$ (2c) with $s := s;t$ $\}$

$$[\,t\,;\,*(s;t) \Rightarrow t\,;\,*(s;t)\,]$$

$\equiv$ $\{$predicate calculus$\}$

true .

As a result, $t\,;\,*(s;t)$ is the strongest solution of $x : [\,t \vee x;s;x \Rightarrow x\,]$.

<u>Remark</u> Had we followed the strategy pointed out in the foregoing Remark, we would have found $*(t;s)\,;\,t$ as the strongest solution. Since strongest solutions are unique, we would thus – on the fly – have established the "Leap Frog Rule":

$$[\,t\,;\,*(s;t) \equiv *(t;s)\,;\,t\,]$$

<u>End</u> of Remark.

<u>Final Remarks</u>

- We have <u>not</u> used the universal disjunctivity of the composition.
- In WF175 ("Playing with dagger and star, i.e. with transitive closures") we could develop a great deal of the regularity calculus without an appeal to the

to the composition's universal disjunctivity either

- The combination of these two experiences may form the basis for a totally different development of the calculus, in which the universal disjunctivity of the composition is encapsulated in – ideally – one single theorem.
- Whether the successful derivation in this note has been a stroke of good luck, or whether it forms an example of a method, remains to be seen.

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