An overlooked formula (from the predicate calculus)

During one of its recent sessions, the ETAC ran into the obligation to prove, for numbers x and y,

$$x \leq y \wedge (y \leq x \vee R) \equiv x \leq y \wedge (x = y \vee R),$$

which we did by disentangling $x = y$ into $x \leq y \wedge y \leq x$, and by distributing $\wedge$ over $\vee$, etcetera.

When Rik van Geldrop saw this, she immediately generalized the equivalence into

(0) $[P \wedge (Q \vee R) \equiv P \wedge ((P \wedge Q) \vee R)]$.

<u>Proof</u>

$$\begin{array}{ll} & P \wedge ((P \wedge Q) \vee R) \\ \equiv & \quad \{\wedge \text{ over } \vee\} \\ & (P \wedge P \wedge Q) \vee (P \wedge R) \\ \equiv & \quad \{\text{idempotence of } \wedge\} \\ & (P \wedge Q) \vee (P \wedge R) \\ \equiv & \quad \{\wedge \text{ over } \vee\} \\ & P \wedge (Q \vee R) \end{array}$$

(<u>End</u> of Proof.)

None of us could remember to have seen the rule being recorded explicitly, let alone that it got a name. Nevertheless, it deserves to be added to our basic repertoire of calculational rules, not withstanding the fact that we have already used it so often in its equivalent form

$$[P \Rightarrow (Q \vee R \equiv (P \wedge Q) \vee R)] ,$$

when "calculating under context".

Rule (0) grants us the opportunity to import or export "context" P, in one fell swoop, into or from any of the disjuncts with which P is conjoined.

Of course we also have the dual

(1) $$[P \vee (Q \wedge R) \equiv P \vee ((P \vee Q) \wedge R)] ,$$

importing or exporting context $\neg P$.

For the time being, we will refer to (0) and (1) as "the context rules".

On behalf of the ETAC,
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PS Rule (0) can be further generalized into

$$\begin{array}{l} [P0 \Rightarrow P1] \\ \quad \Rightarrow \quad [P0 \wedge (Q \vee R) \equiv P0 \wedge ((P1 \wedge Q) \vee R)] , \end{array}$$

but I don't know how useful this is.