Exploiting universal junctivity

We consider a lattice with partial order $\leq$, and in which all suprema and infima exist. Two endofunctions f and g are so-called Galois-connected whenever

$$\langle \forall x,y :: \ f.x \leq y \ \equiv \ x \leq g.y \rangle \ ;$$

function g is called f's upper adjoint and function f is called g's lower adjoint.

It is very well-known that for such functions f and g, f distributes over arbitrary suprema — f is universally $\uparrow$-junctive — and that g distributes over arbitrary infima — g is universally $\downarrow$-junctive — .

It is also well-known that in dealing with such functions, it is always a good idea to write down the so-called cancellation rules

$$\langle \forall x :: \ x \leq g.(f.x) \rangle \quad \text{and}$$

$$\langle \forall y :: \ f.(g.y) \leq y \rangle \ .$$

Furthermore there is the theorem that a universally $\uparrow$-junctive function has an upper Galois adjoint, and that a universally $\downarrow$-junctive function has a lower Galois adjoint. Unfortunately, this theorem seems to reside too much in the backs of our minds, whereas in the elementary theory of Galois-connections it should, we think, be treated as a first-class citizen. The purpose of this note is to create evidence for this.

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We shall prove the following theorem from the relation calculus. For any relation R the two equations

$$X: \ [J \vee X;R \Rightarrow X] \qquad \text{and}$$

$$X: \ [J \vee R;X \Rightarrow X]$$

have the same strongest solution. (It is taken for granted here that each individual equation has a strongest solution indeed.)

Let P and Q be the respective strongest solutions, i.e. — spelled out —

(0a) $\langle \forall X: [J \Rightarrow X] \wedge [X;R \Rightarrow X] : [P \Rightarrow X] \rangle$
(0b) $[J \Rightarrow P]$
(0c) $[P;R \Rightarrow P]$

(1a) $\langle \forall X: [J \Rightarrow X] \wedge [R;X \Rightarrow X] : [Q \Rightarrow X] \rangle$
(1b) $[J \Rightarrow Q]$
(1c) $[R;Q \Rightarrow Q]$.

Then the theorem states: $[P \equiv Q]$.
We prove $[P \Rightarrow Q]$; the remaining $[Q \Rightarrow P]$ follows from duality.

$$
\begin{array}{ll}
 & [P \Rightarrow Q] \\
\Leftarrow & \{ (0a) \text{ with } X := Q \} \\
 & [J \Rightarrow Q] \wedge [Q ; R \Rightarrow Q] \\
\equiv & \{ (1b) \} \\
 & [Q ; R \Rightarrow Q]
\end{array}
$$

At this point both (1b) and (1c) are applicable by using the transitivity of $\Rightarrow$, but both yield hopeless new expressions So we now must seek to massage our expression such that (1a) becomes applicable, i.e. massage it into the form $[Q \Rightarrow \text{something}]$. And at this point it is good to realize that "semi" (composition) is universally disjunctive in both arguments. (In the lattice of predicates, "suprema" are called "disjunctions".)

We now define function f by

$$[f.X \equiv X ; R].$$

Since it is universally disjunctive, it has an upper Galois adjoint, to be called g. So we have

(2) $\quad [f.X \Rightarrow Y] \equiv [X \Rightarrow g.Y] \qquad (\forall X, Y)$.

The corresponding cancellation rules are

(3a) $[X \Rightarrow g.(f.X)]$ $(\forall X)$ and

(3b) $[f.(g.Y) \Rightarrow Y]$ $(\forall Y)$.

And now we are ready to resume our calculation:

$[Q ; R \Rightarrow Q]$

$\equiv$ { definition of f }

$[f.Q \Rightarrow Q]$

$\equiv$ { Galois connection (2) }

$[Q \Rightarrow g.Q]$

$\Leftarrow$ { (1a) with $X := g.Q$ }

$[J \Rightarrow g.Q] \wedge [R ; g.Q \Rightarrow g.Q]$

$\equiv$ { Galois connection (2) on both conjuncts }

$[f.J \Rightarrow Q] \wedge [f.(R ; g.Q) \Rightarrow Q]$

$\equiv$ { definition of f on both conjuncts }

$[J ; R \Rightarrow Q] \wedge [(R ; g.Q) ; R \Rightarrow Q]$

$\equiv$ { first conjunct equivales true, see below; associativity of semi on the second conjunct }

$[R ; (g.Q ; R) \Rightarrow Q]$

$\equiv$ { definition of f }

$[R ; f.(g.Q) \Rightarrow Q]$

$\Leftarrow$ { cancellation rule (3b) with $Y := Q$ }

$[R ; Q \Rightarrow Q]$

$$
\begin{array}{ll}
 & [R ; Q \Rightarrow Q] \\
\equiv & \quad \{ (1c) \} \\
 & \text{true} .
\end{array}
$$

What remains is the (simple) proof of $[J ; R \Rightarrow Q]$:

$$
\begin{array}{ll}
 & J ; R \\
\equiv & \quad \{ \text{identity of semi} \} \\
 & R ; J \\
\Rightarrow & \quad \{ (1b) \text{ and monotonicity of semi} \} \\
 & R ; Q \\
\Rightarrow & \quad \{ (1c) \} \\
 & Q
\end{array}
$$

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Final Remarks

- In the above we intentionally used the neutral and meaningless names f and g for the Galois adjoints, rather than introducing some sort of factor notation.
- In the above proof we intentionally carried out our manipulations in terms of f as much as possible, rather than in terms of its specific instance in this example, such for the sake of simplifying pattern matching.
- In the above proof we intentionally

devoted a step to the associativity of semi: focussing the attention on the "other" parsing is of utmost heuristic importance (regardless of whether the proof is designed/read from top to bottom or from bottom to top!).

- The intended moral of course is, that when a universally junctive function is involved, a bell should ringing: that function has a Galois adjoint.

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