Richard Bird's Typewriter Problem

We consider a typewriter equipped with a TAB-command and with a SPACE-command. These commands are specified as follows:

- whenever the carriage is at position x, a SPACE moves it to position $x+1$
- whenever the carriage is at position x, a TAB moves it to the first multiple of t exceeding x — t given, fixed, and at least 1 —

The question posed by Richard Bird is to come up with an arithmetic expression for the minimum number of commands needed to move the carriage from position m to position n, $0 \le m \le n$. Moreover, he asked how many SPACEs and how many TABs would be needed for this.

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In view of the specification of TAB, we introduce expression $x \underline{\text{red}}\ t$, defined by

$$x \ \underline{\text{red}}\ t \ = \ t * (x \ \underline{\text{div}}\ t) .$$

Then a TAB moves the carriage from position x to position $x \ \underline{\text{red}}\ t + t$.

Being programmers, we decided to design a program computing the minimum number of TABs and SPACEs needed. The program

— discussed below — reads

```
0.      x, tabs := m red t, 0
1.      do x+t ≤ n → x, tabs := x+t, tabs + 1 od
2.      spaces := (n−x)↓(n−m)
```

The first observation is that when a TAB is impossible because it would move the carriage beyond n, SPACEs are obligatory. This is the case at all carriage positions i for which n red $t \leq i$. In the above program, carriage position x is a multiple of t, and upon termination of the repetition

(a) $\quad x = n$ red t

holds, so that at this position SPACEs are obligatory.

The second observation is that at positions i for which $i < n$ red t, a TAB *can* be given and is never worse than a SPACE — $t \geq 1$! —. Therefore a TAB *will* be given at those positions. This is reflected by line 1, because $x + t \leq n \Rightarrow x < n$ red t.

The third observation is that, as far as the number of TABs is concerned, the answer to the problem would not change if m red t instead of m were the initial carriage position. This is reflected by line 0 of the program.

Now we observe that the repetion has as an invariant

(b) $\quad x - t \times tabs = m$ red t

from which we conclude that upon termination

$$
\begin{array}{ll}
 & tabs \\
= & \quad \{ (b) \} \\
 & (x - m \,\underline{red}\, t) / t \\
= & \quad \{ (a) \} \\
 & (n \,\underline{red}\, t - m \,\underline{red}\, t) / t \\
= & \quad \{ \text{definition of } \underline{red} \} \\
 & n \,\underline{div}\, t - m \,\underline{div}\, t \quad .
\end{array}
$$

So the number of TABs is $n \,\underline{div}\, t - m \,\underline{div}\, t$.

In determining the number of SPACEs, we have to be a little bit careful. If, after all the TABs are given, $x \leq m$, i.e. $n \,\underline{red}\, t \leq m$, $n-m$ SPACEs are obligatory, otherwise $n-x$ ($x \leq n$ is an invariant of the repetition). Whence the assignment in line 2. An arithmetic expression for the number of SPACEs is

$$(n \,\underline{mod}\, t) \downarrow (n-m) \quad .$$

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