Playing with $\uparrow$ and $|\cdot|$

This note is just for the record, adding a few more samples to our collection of simple calculational proofs in elementary mathematics. This time we are concerned with the relations between the maximum ($\uparrow$) and the absolute-value function ($|\cdot|$).

A calculationally nice definition of $|\cdot|$ is the following :

(0) $\quad |x| = x \uparrow -x$.

With it —and properties of $\uparrow$, of course— we can prove properties like

(1) $\quad |x| \geq 0$

(2) $\quad |x| = (x \uparrow 0) + (-x \uparrow 0)$

(3) $\quad ||x|| = |x|$

(4) $\quad |x| + |y| = |x+y| \uparrow |x-y|$.

Let me calculate (4), which is a nice property, because the triangular inequality follows from it.

$$
\begin{aligned}
& |x| + |y| \\
= & \quad \{\text{def. } 1.1\} \\
& (x \uparrow -x) + (y \uparrow -y) \\
= & \quad \{+ \text{ over } \uparrow\} \\
& (x+y) \uparrow (x-y) \uparrow (-x+y) \uparrow (-x-y) \\
= & \quad \{\text{reshuffling and def. } 1.1\} \\
& |x+y| \uparrow |x-y| \quad .
\end{aligned}
$$

Now let's deal with some other problems, which came up during a discussion with some colleagues. Here is the first question.

For what values of x and y is it true that

(5) $$x \uparrow y \;=\; \frac{|x+y| + |x-y|}{2} \qquad ?$$

So we start calculating with the right-hand side, finding

$$
\begin{aligned}
& |x+y| + |x-y| \\
= & \quad \{(4)\} \\
& |2 \cdot x| \uparrow |2 \cdot y| \\
= & \quad \{2\cdot \text{ over } |\,| \text{ and } \uparrow\} \\
& 2 \cdot (|x| \uparrow |y|) \quad .
\end{aligned}
$$

Apparently we've proved:

$$(5) \quad \equiv \quad x \uparrow y \;=\; |x| \uparrow |y| \quad ;$$

although the right-hand side of this equivalence is, indeed, a simplification of (5), we want to simplify it a little further still.

$$x \uparrow y \;=\; |x| \uparrow |y|$$

$$\equiv \quad \{\text{exit } 1{\cdot}1\}$$

$$x \uparrow y \;=\; x \uparrow -x \uparrow y \uparrow -y$$

$$\equiv \quad \{\text{reshuffling; using } a = a \uparrow b \;\equiv\; a \geq b\}$$

$$x \uparrow y \;\geq\; -x \uparrow -y$$

$$\equiv \quad \{- \text{ over } \uparrow\}$$

$$x \uparrow y \;\geq\; -(x \downarrow y)$$

$$\equiv \quad \{\text{shunting}\}$$

$$(x \uparrow y) + (x \downarrow y) \;\geq\; 0$$

$$\equiv \quad \{\uparrow/\downarrow \text{ selectors}\}$$

$$x + y \;\geq\; 0 \quad .$$

So we have

$$(6) \quad x \uparrow y = |x| \uparrow |y| \quad \equiv \quad x + y \geq 0 \quad .$$

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We could consider (5) as an – unsuccessful – conjecture of how $\uparrow$ can be expressed in terms of $|\cdot|$. We can do somewhat better, however.

First, there is a property that helps bridging the gap between the two-argument function $\uparrow$ and the single-argument $|\cdot|$:

$$(7) \qquad x \uparrow 0 \; = \; \frac{x + |x|}{2} \quad .$$

And now we calculate :

$$\begin{array}{cl} & x \uparrow y \\ = & \{ +y \text{ over } \uparrow \} \\ & (x-y) \uparrow 0 \;+\; y \\ = & \{ (7) \} \\ & \dfrac{x-y \;+\; |x-y|}{2} \;+\; y \\ = & \{ \text{calc.} \} \\ & \dfrac{x+y \;+\; |x-y|}{2} \quad . \end{array}$$

That's it. Thanks go to Jos Brands & Malo Hautus.

W.H.J. Feijen & A.J.M. van Gasteren
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