An exercise given by Rutger M. Dijkstra

Here is a tiny little exercise that Rutger M. Dijkstra gave us and that we solved in his presence. The exercise is definitely inspired by Rutgers study of Burghard von Kargers "Temporal Logic via Galois Connections". We record it because it could nicely fit in into our forthcoming course on "Calculational Mathematics" (if the mathematical community will ever allow us to give such a course.

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We consider the lattice of monotonic predicate transformers. For (monotonic) predicate transformers f and g, expression $f \leq g$ stands for $\langle \forall b :: [f.b \Rightarrow g.b] \rangle$.

We now are given two fixed monotonic predicate transformers f and g, and two endofunctions (predicate transformer transformers) $\oplus$ and $\ominus$ on our lattice, such that

(0) $\quad [(\oplus g).b \equiv g.(f.b)]$

(1) $\quad \ominus$ is the lower adjoint of $\oplus$, i.e.

$$\ominus g \leq h \equiv g \leq \oplus h .$$

The exercise is to derive an expression for predicate $(\ominus g).b$. The expression should be expressed on "the predicate level", not on "the level of predicate transformers".

<u>Remark</u> In order to make this latter requirement

more precise, Rutger at this point revealed the answer, viz.

$$[(\ominus g).b \;\equiv\; \langle \exists c: [f.c \Rightarrow b] : g.c \rangle \;] ,$$

which was quite a strong hint.
End of Remark.

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Since (1) constitutes our only given about $\ominus$, there is nothing else we can do than to spell out (1) and render it on the predicate level:

(2) $$\langle \forall b :: [(\ominus g).b \Rightarrow h.b] \rangle$$
$$\equiv \quad \langle \forall c :: [g.c \Rightarrow (\oplus h).c] \rangle \quad .$$

Since there is nothing useful we can do to the lefthand side of (2), and we therefore start massaging its righthand side. The overall strategy is that we massage it, by a number of equality preserving steps, into an expression of the form

$$\langle \forall b :: [\text{ some } \underline{\text{monotonic}} \text{ predicate in } b \;\; \Rightarrow h.b] \rangle .$$

Then, thanks to the validity of (2), that monotonic predicate in b is an appropriate expression for $(\ominus g).b$. With this in mind, we observe

$$\langle \forall c :: [g.c \Rightarrow (\oplus h).c] \rangle$$
$$\equiv \qquad \{ \text{ (0), i.e. definition of } \oplus \}$$
$$\langle \forall c :: [g.c \Rightarrow h.(f.c)] \rangle$$

$\equiv$ { in view of our target expression, we have to remove the f as an argument of h. We could do that with one of the one-point rules and rewrite $h.(f.c)$ as $\langle\forall b: [b \equiv f.c] : h.b\rangle$ or as $\langle\exists b: [b \equiv f.c] : h.b\rangle$, but with the monotonicity requirements lurking around the corner, we had better resort to one of the rewrite rules
$[\ p.y \ \equiv \langle\forall b: [y \Rightarrow b] : p.b\rangle\ ]$ or
$[\ p.y \ \equiv \langle\exists b: [b \Rightarrow y] : p.b\rangle\ ]$,
which hold for monotonic p (are even equivalent to p being monotonic). See EWD1147. We choose to rewrite $h.(f.c)$ as $\langle\forall b: [f.c \Rightarrow b] : h.b\rangle$, because in our current expression $(g.c \Rightarrow)$, $[\,]$, and $(\forall c)$ all distribute over universal quantification. }

$$\langle\forall c:: [\ g.c \Rightarrow \langle\forall b: [f.c \Rightarrow b] : h.b\rangle\ ]\ \rangle$$

$\equiv$ { predicate calculus }

$$\langle\forall b:: \langle\forall c: [f.c \Rightarrow b] : [g.c \Rightarrow h.b]\ \rangle$$

(*) $\equiv$ { predicate calculus }{ see Remark below }

$$\langle\forall b:: [\ \langle\exists c: [f.c \Rightarrow b] : g.c\rangle \ \Rightarrow h.b\ ]\ \rangle$$

and here we have arrived at the target shape and we can satisfy (2) by choosing

$$(\ominus g).b \ \equiv \langle\exists c: [f.c \Rightarrow b] : g.c\rangle\ ,$$

which is monotonic in b indeed.

<u>Remark</u> The step marked (*) in the above

calculation seems to be a combination of several smaller steps from the predicate calculus, but as Rutger pointed out, we had better learn to consider that as just one <u>elementary</u> step. He drew an analogy with an absolutely elementary rule from lattice calculus, viz.

$$\begin{array}{l} \langle \forall c : R.c : t.c \leq z \rangle \\ \equiv \\ \langle \sup c : R.c : t.c \rangle \leq z \end{array} ,$$

which indeed is in the foreground of our mind. If we now realize that in the lattice of predicates, $x \leq y$ is rendered as $[x \Rightarrow y]$ and sup is rendered as $\exists$, the step marked ⊗ just <u>is</u> this elementary step from lattice calculus.
<u>End</u> of Remark

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Rutger gave us another exercise. We here record it for our files. Again in the lattice of monotonic predicate transformers, infix operator $\rightarrow$ is given to satisfy

$$f \wedge g \leq h \quad \equiv \quad g \leq (f \rightarrow h) .$$

The exercise is to find an expression for

$$(f \rightarrow h).b$$

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