A bagatelle on extreme solutions

(Presumably, the little theorems in this note are well-known and can be found in some books and in the heads of many people. I have never encountered them there and I think they are nice enough to be dealt with in isolation.)

We consider a poset with partial order $\leq$. Furthermore, $\uparrow$ (the supremum) and $\downarrow$ (the infimum) are defined for each pair of elements. We now have the following theorem

Theorem Let p, q, and r be such that

(0) p is the least solution of $x : b.x$
(1) q is the least solution of $x : c.x$
(2) r is the least solution of $x : b.x \vee c.x$,

then $r = p \downarrow q$

Proof We spell out the givens (0), (1), and (2):

(0a) $b.x \Rightarrow p \leq x$, $(\forall x)$
(0b) $b.p$

(1a) $c.x \Rightarrow q \leq x$, $(\forall x)$
(1b) $c.q$

(2a) $(b.x \Rightarrow r \leq x) \wedge (c.x \Rightarrow r \leq x)$, $(\forall x)$
(2b) $b.r \vee c.r$

Now,

$$\begin{array}{ll} & r \leq p \downarrow q \\ \equiv & \quad \{\text{def. of } \downarrow\} \\ & r \leq p \wedge r \leq q \\ \Leftarrow & \quad \{(2a)\} \\ & b.p \wedge c.q \\ \equiv & \quad \{(0b), (1b)\} \\ & \text{true} \end{array}$$

$$\begin{array}{ll} & p \downarrow q \leq r \\ \Leftarrow & \quad \{\text{property of } \downarrow\} \\ & p \leq r \vee q \leq r \\ \Leftarrow & \quad \{(0a), (1a)\} \\ & b.r \vee c.r \\ \equiv & \quad \{(2b)\} \\ & \text{true} \end{array}$$

End of Proof.

By flipping $\leq$ and $\geq$, we also flip least and greatest, and $\downarrow$ and $\uparrow$, and thus we obtain the so-called dual theorem

<u>Theorem</u> Let p, q, and r be such that

p is the greatest solution of $x : b.x$
q is the greatest solution of $x : c.x$
r is the greatest solution of $x : b.x \vee c.x$,

then $r = p \uparrow q$.

<u>End</u>

For the conjunctive equation $x : b.x \wedge c.x$ the results are less nice, but nice enough to be recorded

<u>Theorem</u> Let p, q, and r be such that

p is the least solution of $x : b.x$
q is the least solution of $x : c.x$
r is the least solution of $x : b.x \wedge c.x$

then (i) $p \uparrow q \leq r$ and (hence) $p \downarrow q \leq r$
(ii) $c.p \vee b.q \Rightarrow r = p \uparrow q$
(iii) $c.p \wedge b.q \Rightarrow r = p \downarrow q$

and (iv) $r = p \downarrow q \Rightarrow p = q$
(v) $c.p \Rightarrow r = p$
(vi) $b.q \Rightarrow r = q$

<u>End</u>

The straightforward, yet nice, little proofs are left to the reader.

I expect more little theorems from "extremity theory" to emerge in the near future.

Sterksel, WHJ Feijen.
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