$\pi f^2 = \pi f$, courtesy Jaap van der Woude and Henk Doornbos

We consider a monotonic function f on a partially ordered set. Furthermore the poset is such that for any two elements the infimum ($\downarrow$) exists. The theorem to be proved is

f has a least prefix-point
$\Rightarrow$
f^2 has a least prefix-point .

<u>Proof</u> Prefix-points are unique. Let πf be *the* prefix-point of f, i.e.

(0a) $\quad f.x \leq x \ \Rightarrow \ \pi f \leq x \qquad (\forall x)$

(0b) $\quad f.\pi f \leq \pi f$.

We have to prove the existence of a least-prefix-point of f^2, i.e. we have to prove the existence of a p such that

(1a) $\quad f^2.x \leq x \ \Rightarrow \ p \leq x \qquad (\forall x)$

(1b) $\quad f^2.p \leq p$.

We start to investigate what is involved in the wanted validity of (1b).

$f^2.p \leq p$
$\quad$ { of the two givens (0a) and (0b), the only one that more or less matches $f^2.p \leq p$ is (0b). But then we have to get rid

of an occurrence of f in f^2. Our means towards such an elimination are Leibniz and monotonicity, but these require an f at both sides. Hence we propose the availability of a q such that

$\Leftarrow$

• $f.q \le p$ } { transitivity of $\le$ }

$f^2.p \le f.q$

$\Leftarrow$ { f is monotonic }

$f.p \le q$

$\equiv$ { • $f.p \le q$ }

true .

In order to satisfy both $f.q \le p$ and $f.p \le q$ we have absolutely no choice: $p = \pi f$ and $q = \pi f$ — see (0b) — .

After the above the only candidate left for p is πf, in terms of which our remaining proof obligation (1a) reads

$$f^2.x \le x \;\Rightarrow\; \pi f \le x \; . \qquad (\forall x)$$

A proof of this may have the form

$\pi f \le x$

$\Leftarrow$ { (0a) }

$f x \le x$

$\Leftarrow$ { ??? }

$f^2.x \le x$.

There is however no way in which we can validate the step marked ???. (There even is no way: Jaap van der Woude showed a counterexample.)

So we have to proceed more carefully and introduce an extra f so as to create an f^2. Here the existence of $\downarrow$ enters the picture

$$
\begin{array}{ll}
 & \pi f \leq x \\
\Leftarrow & \quad \{\text{Big Trick}\}\{\ x \downarrow f.x \leq x\ \} \\
 & \pi f \leq x \downarrow f.x \\
\Leftarrow & \quad \{\ (0a) \text{ with } x := x \downarrow f.x\ \} \\
 & f.(x \downarrow f.x) \leq x \downarrow f.x \\
\Leftarrow & \quad \{\ f.(x \downarrow f.x) \leq f.x \downarrow f.(f.x)\ ,\ \text{from } f\text{'s} \\
 & \quad\quad \text{monotonicity}\} \\
 & f.x \downarrow f.(f.x) \leq x \downarrow f.x \\
\Leftarrow & \quad \{\ \downarrow \text{ is monotonic}\}\ \{\ f \circ f = f^2 \} \\
 & f^2.x \leq x \quad .
\end{array}
$$

* * *

It is the Big Trick that triggered the writing of this note. It was Henk Doornbos who drew my attention to the fact that the Big Trick is not a trick at all since this calculational move pops up every so often in fixed-point calculations.

* * *

Jaap van de Woude's counterexample for $f.x \leq x \Leftarrow f^2.x \leq x \quad (\forall x)$ is along the following lines. Take the natural numbers with ordering

$$0 \geq 2 \geq 4 \geq 6 \geq \cdots$$
$$0 \geq 1 \geq 3 \geq 5 \geq \cdots ,$$

and no relation between the positive even and positive odd numbers. Take for f: $f.0 = 0$, $f.(2k+2) = 2k+1$, $f.(2k+1) = 2k+2$. Then $f^2 = Id$, and $f \neq Id$.

WHJ Feijen

3 November 1995