A very beginning of lattice theory

Let's start at the very beginning.
A very good place to start.

Julie Andrews in *The Sound of Music*

For a large part, mathematics consists of exploring concepts and of investigating and proving their properties. The art of proving plays a major rôle in this game. Since the advent of modern computing science, it has become clear that in many branches of elementary mathematics, proofs can be beneficially rendered in a calculational format. The benefits comprise greater precision and lucidity —without loss of concision—, an enhanced view on how to separate one's concerns, and hence an improved economy of thought. Unfortunately, most textbooks on elementary mathematical issues have not (yet) adopted such a calculational style, so that yet another generation of young people will receive a mathematical education without having experienced the joy and usefulness of calculating. And this is a pity.

The purpose of this note is to transmit some of the flavour of calculation. We have selected a topic from the very beginning of lattice theory and we intend to present a treatment that can be read, understood, and hopefully enjoyed by a reasonable university freshman.

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Our universe of discourse will be some fixed, anonymous set of things on which a binary relation $\leq$ (" at most") is defined. This relation we postulate to be

- reflexive, i.e. $x \leq x$ $(\forall x)$
- antisymmetric, i.e. $x \leq y \land y \leq x \Rightarrow x = y$ $(\forall x, y)$

Remark In the standard literature we usually find the additional postulate that $\leq$ is

transitive, i.e. $x \leq y \land y \leq z \Rightarrow x \leq z$ $(\forall x, y, z)$

For the time being though, we do not need the transitivity of $\leq$. Therefore, we do not introduce it now. And apart from that, it will — as we

shall see — enter the picture in a totally different way.
End Remark .

Equality of things is a very important concept to have. It is as important as the notion of a function. Equality and functions are at the heart of mathematics, and they are beautifully related by the

Rule of Leibniz

For any function f[1], $x = y \;\Rightarrow\; f.x = f.y$.

End

The two postulates that we have of $\leq$ do not reveal very much about equality; only the antisymmetry mentions it. Therefore, the first thing to do is to collect some more facts concerning equality. The most common one is the

Rule of Mutual Inequality

$$x = y \;\equiv\; x \leq y \,\wedge\, y \leq x \quad .$$

End

It is an immediate restatement of $\leq$'s reflexivity and antisymmetry.

A very useful but less common statement about equality is the so-called

Rule of Indirect Equality

$$x = y \;\equiv\; \langle\, \forall z :: \; z \leq x \equiv z \leq y \,\rangle \quad .$$

End

Let us prove it. We prove it by mutual implication. (In our jargon we refer to the implication $LHS \Rightarrow RHS$ from left to right by "ping" and to $RHS \Rightarrow LHS$ by "pong".)

Proof of ping

$$
\begin{array}{ll}
 & x = y \;\Rightarrow\; \langle\, \forall z :: \; z \leq x \equiv z \leq y \,\rangle \\
\equiv & \qquad \{\ (P\Rightarrow) \text{ distributes over } \forall\ \} \\
 & \langle\, \forall z :: \; x = y \;\Rightarrow\; (z \leq x \equiv z \leq y) \,\rangle \\
\equiv & \qquad \{\ \text{Rule of Leibniz, see below}\ \} \\
 & \mathit{true}
\end{array}
$$

The function f involved in this application is the boolean function given by $f.a \;\equiv\; z \leq a$.

[1] We denote function application by an infix dot.

End

Proof of pong We have to prove

$$\langle \forall z :: z \leq x \equiv z \leq y \rangle \Rightarrow x = y \quad ,$$

and we do so by setting up a weakening chain of predicates that begins with the antecedent and ends with the consequent. Notice that in this chain we will quite likely have to refer to the antisymmetry of $\leq$ because this is the only property of $\leq$ that mentions the =-symbol; and we have not used it in the ping-part yet. (The latter remark is a very simple example of the kind of bookkeeping that has proven to be very useful in proof design.) Here is the chain

$$\begin{array}{ll} & \langle \forall z :: z \leq x \equiv z \leq y \rangle \\ \Rightarrow & \quad \{ \text{ instantiate with } z := x \text{ and with } z := y \ \} \\ & (x \leq x \equiv x \leq y) \ \wedge \ (y \leq x \equiv y \leq y) \\ \equiv & \quad \{ \text{ reflexivity of } \leq \ \} \\ & x \leq y \ \wedge \ y \leq x \\ \Rightarrow & \quad \{ \text{ antisymmetry of } \leq \ \} \\ & x = y \end{array}$$

Notice that the first step —the instantiation— is not brilliant at all: the first line contains symbol $\forall$ and the target line does not, so that somewhere along the way we must eliminate $\forall$. In fact, about the only rule from the predicate calculus with which one can eliminate the universal quantifier $\forall$, is the rule of instantiation. Once we are aware of this, the step is no longer a surprise. Furthermore, there is not much we can instantiate z with, viz. just x and y ; and we did both in order to make the next line as strong as possible, which is beneficial if one has to construct a weakening chain.
End Proof of pong .

The rule of Indirect Equality has a companion, also called the Rule of Indirect Equality. It reads

$$x = y \ \equiv \ \langle \forall z :: x \leq z \equiv y \leq z \rangle \quad .$$

The difference is the side of the $\leq$ -symbol at which x and y reside. Which of the two is to be used depends on the particular application.

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So much for $\leq$ and for $=$ in our universe. We now enter lattice theory by postulating that in our universe the equation in p

$$p:\langle \forall z::\ p\leq z\ \equiv\ x\leq z\ \wedge\ y\leq z\ \rangle$$

has, for each x and y, at least one solution. (The inexperienced reader should not feel daunted here: in case our universe is just the universe of real numbers with the usual $\leq$-relation, the maximum of x and y may be recognized as a good candidate for p.)

The first thing we do is to show that the equation has at most one solution. This is done by showing $p=q$, for p and q solutions of the equation. Here, one of the rules of Indirect Equality comes in handy: for any z, we have

$$\begin{array}{ll} & p\leq z \\ \equiv & \quad \{\ p \text{ is a solution } \} \\ & x\leq z\ \wedge\ y\leq z \\ \equiv & \quad \{\ q \text{ is a solution } \} \\ & q\leq z, \end{array}$$

and, hence, $p=q$. So our equation has exactly one solution for each x and y. Therefore, that solution is a function of x and y, which we propose to denote by $x\uparrow y$ (x "up" y). In summary, we have the beautiful

(0) $$x\uparrow y\leq z\ \equiv\ x\leq z\ \wedge\ y\leq z \qquad (\forall x,y,z)$$

(In the standard literature we find $\uparrow$ under entries like "sup" or "join" or "lub".)

Examples

- A well-known instance of (0) can be found in set theory. If we take set inclusion $\subseteq$ as an instance of $\leq$ —it is reflexive and antisymmetric!— , set union $\cup$ is the corresponding $\uparrow$. Indeed, we have for all sets x, y, and z,

  $$x\cup y\subseteq z\ \equiv\ x\subseteq z\ \wedge\ y\subseteq z\ .$$

- Also, if we take set containment $\supseteq$ for $\leq$, set intersection is the corresponding $\uparrow$. Indeed,

  $$x\cap y\supseteq z\ \equiv\ x\supseteq z\ \wedge\ y\supseteq z\ .$$

- Another well-known instance is in the predicate calculus where we have

  $$[\ x\vee y\Rightarrow z\ ]\ \equiv\ [\ x\Rightarrow z\ ]\wedge[\ y\Rightarrow z\ ]\ , \quad \text{and}$$

$$[\, x \wedge y \Leftarrow z \,] \;\equiv\; [\, x \Leftarrow z \,] \wedge [\, y \Leftarrow z \,] \;.$$

- From number theory we know the reflexive, antisymmetric relation denoted $\mid$ ("divides"). Now, the least common multiple of x and y can see the light via

  $$(x \;\mathsf{lcm}\; y) \mid z \;\;\equiv\;\; x \mid z \;\wedge\; y \mid z \quad ,$$

  and the greatest common divisor of x and y by

  $$z \mid (x \;\mathsf{gcd}\; y) \;\;\equiv\;\; z \mid x \wedge z \mid y \quad .$$

  Both are instances of (0). (How?)

- But probably the best-known instance of (0) is when we take for $\leq$ the usual order between numbers. Then $\uparrow$ is the familiar maximum operator. We will return to this later.

End Examples .

Now let us investigate (0). We can rather straightforwardly deduce from it that

- $\uparrow$ is idempotent, i.e. $\quad x \uparrow x \;=\; x$
- $\uparrow$ is symmetric, i.e. $\quad x \uparrow y \;=\; y \uparrow x$
- $\uparrow$ is associative, i.e. $\quad x \uparrow (y \uparrow z) \;=\; (x \uparrow y) \uparrow z \quad .$

Let us prove the symmetry. We appeal to Indirect Equality :

$$
\begin{array}{ll}
 & x \uparrow y \leq z \\
\equiv & \qquad \{ \;\; (0) \;\; \} \\
 & x \leq z \;\wedge\; y \leq z \\
\equiv & \qquad \{ \;\; \wedge \text{ is symmetric } \;\; \} \\
 & y \leq z \;\wedge\; x \leq z \\
\equiv & \qquad \{ \;\; (0) \text{ with } x\,,y \;:=\; y\,,x \;\; \} \\
 & y \uparrow x \;\leq\; z,
\end{array}
$$

and the conclusion follows. From this proof we see that $\uparrow$ inherits its symmetry from $\wedge$. The same holds for $\uparrow$'s idempotence and $\uparrow$'s associativity, as the reader may verify.

The next thing we do with (0) is to study it for some simple instantiations. For instantiation $z := y$ we find

$$
\begin{array}{ll}
 & x \uparrow y \;\leq\; y \\
\equiv & \qquad \{ \;\; (0) \;\; \} \\
 & x \leq y \;\;\wedge\;\; y \leq y
\end{array}
$$

$\equiv$ { $\leq$ is reflexive }

$x \leq y$.

Thus we have derived the

Rule of Absorption

$x \uparrow y \leq y \equiv x \leq y$

End

Next, from (0) with $z := x \uparrow y$, we find the

Rule of Expansion

$y \leq x \uparrow y$

End

Using Mutual Inequality, we can combine the rules of Absorption and Expansion into

(1) $x \uparrow y = y \equiv x \leq y$ $(\forall x, y)$

Remark Almost every established treatment of lattice theory starts from (1), but that is not nearly as nice as the treatment given here, because the pleasing symmetry exhibited by (0) is completely hidden.
End

So much for some simple instantiations of (0).

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Now the time has come to prove the beautiful[2]

Theorem 1 For reflexive and antisymmetric $\leq$, and for $\uparrow$ as defined by (0), we have that

$\leq$ is transitive .

Proof We have to prove that for all x , y , and z

$$x \leq y \wedge y \leq z \Rightarrow x \leq z \ .$$

Using (1), this we can rewrite as

$$x \uparrow y = y \wedge y \uparrow z = z \Rightarrow x \uparrow z = z \ ,$$

and we shall prove this latter by showing the consequent — $x \uparrow z = z$ — thereby using the antecedent — $x \uparrow y = x \wedge y \uparrow z = z$ — :

[2] We owe this theorem to Edsger W. Dijkstra. It seems to be not generally known to lattice theorists.

$x \uparrow z$

$=$ { since $y \uparrow z = z$, from the antecedent }

$x \uparrow (y \uparrow z)$

$=$ { $\uparrow$ is associative }

$(x \uparrow y) \uparrow z$

$=$ { since $x \uparrow y = y$, from the antecedent }

$y \uparrow z$

$=$ { since $y \uparrow z = z$, from the antecedent }

z .

And we are done. (We ask the reader to notice that each individual step in the above calculation is almost forced upon us. This is a very typical characteristic of many a calculation.)
□

Now that we have obtained the transitivity of $\leq$, we shall feel free to use it. For the sake of completeness we mention that a reflexive, antisymmetric, and transitive relation is commonly called a *partial order*, and that a universe equipped with a partial order is called a *partially ordered set* —a "poset" for short—.

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Definition (0) of $\uparrow$ tells us when $x \uparrow y \leq z$. We now may ask when $z \leq x \uparrow y$. We leave to the reader to verify that

(2) $z \leq x \uparrow y \Leftarrow z \leq x \vee z \leq y$,

and we investigate the converse :

$z \leq x \uparrow y \Rightarrow z \leq x \vee z \leq y$

$\equiv$ { predicate calculus }

$(z \leq x \uparrow y \Rightarrow z \leq x) \vee (z \leq x \uparrow y \Rightarrow z \leq y)$

$\Leftarrow$ { $\leq$ is transitive }

$x \uparrow y \leq x \vee x \uparrow y \leq y$

$\equiv$ { Rule of Absorption, twice }

$y \leq x \vee x \leq y$.

For this last line to be valid for any x and y , we require that $\leq$ be a so-called linear or total order: by definition a total order is a partial order

with the additional property that, for all x and y , $x \leq y \lor y \leq x$.
So, in combination with (2) we find

(3) for $\leq$ a total order,
$z \leq x \uparrow y \equiv z \leq x \lor z \leq y$.

Furthermore, we deduce from (1) that

(4) for $\leq$ a total order, operator $\uparrow$ as defined by (0) satisfies
$x \uparrow y = x \lor x \uparrow y = y$.
(In words : $\uparrow$ is a selector.)

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From here, we can proceed in many different directions. After all, lattice theory is a huge mathematical terrain, with many ins and outs. We conclude this introduction by confronting $\uparrow$ with other functions.

We consider functions from and to our anonymous universe. For f such a function we have, by definition,

- f is monotonic $\equiv \langle \forall x, y :: x \leq y \Rightarrow f.x \leq f.y \rangle$.
- f distributes over $\uparrow$ $\equiv \langle \forall x, y :: f.(x \uparrow y) = f.x \uparrow f.y \rangle$.

We can now formulate the well-known, yet beautiful, theorem

(5) f distributes over $\uparrow$ $\Rightarrow$ f is monotonic .

Proof For any x and y , we observe

$f.x \leq f.y$

$\equiv$ { (1) }

$f.x \uparrow f.y = f.y$

$\equiv$ { f distributes over $\uparrow$ }

$f.(x \uparrow y) = f.y$

$\Leftarrow$ { Leibniz's Rule }

$x \uparrow y = y$

$\equiv$ { (1) }

$x \leq y$,

and the result follows from the outer two lines.
End

Small Intermezzo (on proof design)

We would like to draw the reader's attention to the fact that the above proof —no matter how simple it is— displays a great economy of thought. Let us analyze it in some detail. Given that f distributes over $\uparrow$, we have to construct a calculation of the form

$$f.x \leq f.y \ \ldots \Leftarrow \ldots \ x \leq y \quad .$$

Right at the outset we can argue that such a calculation will require at least four steps, viz.

- a step to introduce symbol $\uparrow$, in order to be able to exploit the given about f
- a step in which the given about f is actually used
- a step to eliminate symbol $\uparrow$ again, because the target line $x \leq y$ does not mention it
- a step to eliminate symbol f , for which Leibniz's Rule is our only means so far.

Our proof contains precisely (these) four steps, so it cannot be shortened. In fact, it was designed with these four considerations in mind. When we wrote above "no matter how simple it is", this may have sounded paradoxical, but it isn't. On the contrary, the proof derives its simplicity from the consciously considered shapes of the formulae and from the manipulative possibilities available. Nowadays, many more proofs can be and are being designed following such a procedure.
End Small Intermezzo .

A direct consequence of (5) concerns monotonicity properties of $\uparrow$. Because function f defined by $f.x = c \uparrow x$, for whatever c , distributes over $\uparrow$ —as the reader may verify—, theorem (5) tells us that $\uparrow$ is monotonic in its second argument. Since $\uparrow$ is symmetric, we therefore have

(6) $\uparrow$ is monotonic in both arguments.

What about the converse of (5)? Does it hold as well? In order to find out, we try to prove

$$f.(x \uparrow y) \ = \ f.x \uparrow f.y$$

on the assumption that f is monotonic. We do this by Mutual Inequality:

$$
\begin{array}{ll}
 & f.x \uparrow f.y \ \leq \ f.(x \uparrow y) \\
\equiv & \qquad \{ \text{ definition of } \uparrow \text{, see (0) } \} \\
 & f.x \leq f.(x \uparrow y) \quad \wedge \quad f.y \leq f.(x \uparrow y)
\end{array}
$$

$\Leftarrow \qquad \{$ monotonicity of f , twice $\}$

$x \le x \uparrow y \ \wedge \ y \le x \uparrow y$

$\equiv \qquad \{$ Rule of Expansion, twice $\}$

$true$,

$f.(x \uparrow y) \ \le \ f.x \uparrow f.y$

$\Leftarrow \qquad \{$ (2) $\}$

$f.(x \uparrow y) \le f.x \ \vee \ f.(x \uparrow y) \le f.y$

$\Leftarrow \qquad \{$ monotonicity of f , twice $\}$

$x \uparrow y \le x \ \vee \ x \uparrow y \le y$

$\equiv \qquad \{$ Rule of Absorption, twice $\}$

$y \le x \ \vee \ x \le y$,

and the validity of this last line requires $\le$ to be total. Thus, we have derived, in combination with (5),

(7) for $\le$ a total order,,
f is monotonic $\equiv$ f distributes over $\uparrow$.

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In lattice theory, one always introduces a companion to $\uparrow$; it is $\downarrow$ ("down"). (In the standard literature we find $\downarrow$ under entries like "inf", or "meet", or "glb".) It sees the light via

(8) $z \le x \downarrow y \ \equiv \ z \le x \ \wedge \ z \le y \qquad (\forall x, y, z)$,

i.e. in a way that is very similar to (0). It has very similar —dual— properties to $\uparrow$. In fact, it has the same properties if we simply flip $\le$ into $\ge$, and $\uparrow$ into $\downarrow$: just compare (0) and (8). With this symbol dynamics in mind the companion properties for $\downarrow$ come for free. We mention

- $\downarrow$ is idempotent, symmetric, and associative
- $y \le x \downarrow y \ \equiv \ y \le x$ Absorption
- $x \downarrow y \ \le \ y$ Contraction (= the dual of Expansion)
- $x \downarrow y = y \ \equiv \ y \le x$
- $x \downarrow y \ \le \ z \ \Leftarrow \ x \le z \ \vee \ y \le x$
- $\downarrow$ is monotonic in both arguments
- etcetera .

Of course, we can now also investigate formulae containing both $\uparrow$ and $\downarrow$. We mention

$$x \downarrow (x \uparrow y) = x \quad , \quad x \uparrow (x \downarrow y) = x \quad , \quad \text{and}$$

$$x \downarrow y = x \ \equiv \ x \uparrow y = y \quad .$$

The proofs are left as exercises. We will not continue these investigations now.

In case we take for $\leq$ the usual order between real numbers, $\downarrow$ is the familiar minimum operator.

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Let us, to conclude this story, consider the real numbers with the usual order $\leq$. This is a total order. The foregoing little theory now grants us quite a number of useful arithmetical results.

- In order to find out which part of the $(x\,,y)$-plane satisfies $x \uparrow y \leq x+y$, we simply calculate :

  $$\begin{array}{ll} & x \uparrow y \ \leq \ x+y \\ \equiv & \qquad \{ \ \text{definition of } f \ \ \} \\ & x \leq x+y \ \wedge \ y \leq x+y \\ \equiv & \qquad \{ \ \text{arithmetic} \ \} \\ & 0 \leq y \ \wedge \ 0 \leq x \quad . \end{array}$$

  So the answer is: the first quadrant.
  (Ask one of your colleagues or students to solve this little problem, and observe how he does it. This could be a very instructive experiment.)

- Since function f , defined by $f.x = c+x$, is monotonic, we infer from (7):

  addition distributes over the maximum.

- Likewise, multiplication with a nonnegative number distributes over the maximum.

- And also

  $$2^{x \uparrow y} = 2^x \uparrow 2^y \quad , \quad \text{and}$$

  $$(x \uparrow y)^2 = x^2 \uparrow y^2 \quad (\text{for } x\,,y \geq 0\,,0) \quad , \quad \text{and}$$

$$z \downarrow (x \uparrow y) = (z \downarrow x) \uparrow (z \downarrow y) \quad .$$

- And now the reader should prove —with a minimal amount of case analysis—

$$x^2 \downarrow y^2 \ \leq \ x{*}y \ \ \Leftarrow \ \ 0 \leq x{*}y \quad .$$

- Perhaps, we can also learn to handle absolute values more readily, because we have $|\,x\,| \ = x \uparrow -x$. Try to use it to prove the triangular inequality

$$|\,x{+}y\,| \ \leq \ |\,x\,| + |\,y\,| \quad .$$

- Etcetera.

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This really was the beginning of lattice theory. Was it difficult? We hope that most of our readers will say: no! We believe that elementary lattice theory —which goes beyond this note— can and should be taught to reasonable freshmen or, in any case, to sophomores, of computing science and mathematics alike. Many of our colleagues, world-wide, especially computing science colleagues, will shudder at the thought, because lattice theory is regarded far too abstract to be useful or to be teachable to the average student. And abstract stands for frightening, doesn't it? We really must disagree with such a point of view, because —as we tried to show— the game is completely under control by the use of a modest repertoire of simple calculational rules. It is the peaceful calculational style which does away with the fear for abstract things. And also, it is the peaceful calculational style which lets the subject matter sink in much more profoundly than would have been the case otherwise. If still in doubt, remember Newton and Leibniz: they took away the deep difficulties attending the notions of limits and derivatives by ...proposing a symbolism to denote them and a set of formula rewrite rules to manipulate and ...to master them. By now these notions are high-school topics.