$\exists\forall \Rightarrow \forall\exists$

A very well-known theorem from the predicate calculus is

$$(0)\quad [\langle \exists x : R.x : \langle \forall y : S.y : t.x.y\rangle\rangle \Rightarrow \langle \forall y : S.y : \langle \exists x : R.x : t.x.y\rangle\rangle]$$

These days I wanted to reconstruct the very concise proof of this theorem that Carroll Morgan taught me a number of years ago. But I had some trouble in reconstructing it, presumably because I never thought about the heuristics of that design. Hence this note. (Meanwhile, I encountered an equally concise companion to Carroll's proof.)

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Theorem (0) has the shape $\exists\forall \Rightarrow \forall\exists$. Since we have no rule for interchanging quantifiers, the transformation from $\exists\forall$ to $\forall\exists$ will require at least one intermediate expression. The simplest transformations with one intermediate expression have the form

$\exists\forall$
$\Rightarrow$ {eliminate $\forall$}
$\exists$
$\Rightarrow$ {introduce $\forall$}
$\forall\exists$

(1a)

$\exists\forall$
$\Rightarrow$ {eliminate $\exists$}
$\forall$
$\Rightarrow$ {introduce $\exists$}
$\forall\exists$

(1b)

$$\begin{array}{ll} & \exists\forall \\ \Rightarrow & \{\text{introduce } \forall\} \\ & \forall\exists\forall \\ \Rightarrow & \{\text{eliminate } \forall\} \\ & \forall\exists \end{array}$$

(2a)

$$\begin{array}{ll} & \exists\forall \\ \Rightarrow & \{\text{introduce } \exists\} \\ & \exists\forall\exists \\ \Rightarrow & \{\text{eliminate } \exists\} \\ & \forall\exists \end{array}$$

(2b)

Now, we are led to consider rules for the introduction and elimination of quantifiers. Of course, there are the one-point-rules but they are of no use in our current example, since in (0) the ranges R and S are arbitrary.
Next, of course, we have the rules of instantiation

(3a) $[\langle \forall z : r.z : f.z \rangle \Rightarrow f.y]$, provided $r.y$ [sic]

(3b) $[f.y \Rightarrow \langle \exists z : r.z : f.z \rangle]$, provided $r.y$.

In a weakening chain, as depicted in each of our four schemes, (3a) can be used to eliminate $\forall$ and (3b) to introduce $\exists$. So, we still have to think of rules for introducing $\forall$ and eliminating $\exists$. The simplest we can think of are the anonymous (alas)

(4a) $[\, Q \Rightarrow \langle \forall z :: Q \rangle \,]$ for fresh z
and
(4b) $[\, \langle \exists z :: Q \rangle \Rightarrow Q \,]$ for any range .

Now, in principle, we have the ingredients to implement our schemes. Unfortunately, schemes (1) —which are the simpler— drop out: in our example theorem the elimination of $\forall$ in (1a) requires an instance in the range S, but such an instance is not in stock. And for the same reason the introduction of $\exists$ in (1b) is bound to fail.

So, we are left with (2). Let us first look at (2a). Again, the elimination of $\forall$ has the proviso that an instance in the range S has to be available. Fortunately, the rule for $\forall$-introduction offers us complete freedom in choosing a range, and it is this range which should provide the instance of S that is needed. Thus, we arrive at Carroll Morgan's proof:

$\langle \exists x : R.x : \langle \forall y : S.y : t.x.y \rangle \rangle$

$\Rightarrow$ { (4a) with Q := "above expression" }

$\langle \forall z : S.z : \langle \exists x : R.x : \langle \forall y : S.y : t.x.y \rangle \rangle \rangle$

$\Rightarrow$ { (3a), i.e. instantiate the inner $\forall$ with $y := z$ }

$\langle \forall z : S.z : \langle \exists x : R.x : t.x.z \rangle \rangle$

$=$ { dummy renaming }

$\langle \forall y : S.y : \langle \exists x : R.x : t.x.y \rangle \rangle$

(The companion proof is obtained by using (3b) and (4b) in scheme (2b), and is left to the reader.)

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Predicates can be rendered as special kinds of relations (from relational calculus). Over lunch, I asked some Backhouse Boys — who are my closest colleagues and experts in the relational calculus — how they would prove this theorem relationally. We and they found that this would not be a walk-over, because something like the Cone Rule — a rule revealing the existence of points — would be needed. Moreover, the translation of theorem (0) into a relational format yielded expressions which were not too friendly to relational manipulation. This is no blame on the Backhouse Boys, nor on the relational calculus, but it is good to know that "going pointless" not just has its charm, but its price as well.

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