A theorem on the monotonicity of extreme solutions

We consider a universe we a partial order $\leq$. We also consider equation

$$x: \ [B.x.y]$$,

of which we assume that for any y it has a greatest solution, to be denoted $g.y$. By definition, $g.y$ has the following properties:

(0) $[B.x.y] \Rightarrow x \leq g.y$ (for all x)

— extremity of $g.y$ —

(1) $[B.(g.y).y]$.

Now the theorem to be shown is

" B is weakening in its second argument
$\Rightarrow$
g is monotonic with respect to $\leq$ "

<u>Proof</u> For any p, q we observe

$$
\begin{array}{ll}
 & g.p \leq g.q \\
\Leftarrow & \quad \{\text{extremity of } g.q \quad \text{— see (0) —}\} \\
 & [B.(g.p).q] \\
\Leftarrow & \quad \{B \text{ is weakening in } 2^{nd} \text{ argument}\} \\
 & [B.(g.p).p] \ \wedge \ p \leq q \\
\equiv & \quad \{(1) \text{ with } y := p\} \\
 & p \leq q \ .
\end{array}
$$

(<u>End</u> of Proof.)

(Of we have a brother theorem for smallest solutions, and of course there is nothing special about the *second* argument. But we don't bother about this here.)

I could not find the theorem in the book of Dijkstra and Scholten; hence this note. The theorem is, however, well-known in case equation $x: [B.x.y]$ is a fixed-point equation like

$$x:\ [\, x \leq f.x.y \,] \qquad \text{for monotonic } f.$$

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