Re EWD1170 : " Equilateral triangles and rectangular grids "

From the first paragraph of EWD1170 we quote the problem: " "Does there exist an equilateral triangle whose vertices have integer (orthogonal Cartesian) coordinates?"." From that same paragraph we also quote its last sentence: " I urge the reader to think about the problem before reading on.". And that is what I did. This note records my solution, which was constructed without pencil and paper, and which is completely different from EWD's.

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My first concern was what could be said about a line segment connecting two grid points. With x the length of that segment — its length being its only relevant aspect — we conclude that x^2 is natural (integer and ≥ 0).

Next, is there something we can relate x^2 to? Yes, to area. An equilateral triangle with sides x has area $\frac{1}{4}x^2\sqrt{3}$.

Next, is there something special about triangles whose vertices are grid points? Yes, their area is a natural multiple of $\frac{1}{2}$ (see Appendix, part A).

So, for our target equilateral triangle with sides of length x, we have to solve equation

$$x, y : \ \tfrac{1}{4} x^2\sqrt{3} = \tfrac{1}{2} y ,$$

for naturals x and y Or, simplified,

$$(*) \qquad x, y : \ x^2\sqrt{3} = 2y .$$

However, equation

$$z, y : \ z\sqrt{3} = y$$

has $(0,0)$ as its <u>only</u> natural solution (see Appendix, part B). As a result $(*)$ has $(0,0)$ as its only natural solution. Hence the only equilateral triangle whose vertices are gridpoints is the degenerated one with size zero.

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Appendix

Part A

Please allow me a little bit of linear algebra. The area of a triangle with vertices in grid points is half the area of a parallelogram with its vertices in grid points. The area of the latter equals the determinant of a matrix with just integer elements.

Part B

Let z and y be naturals such that $z\sqrt{3} = y$. Then $3z^2 = y^2$, hence y^2 is a multiple of 3, hence y is a multiple of

3, hence y^2 is a multiple of 9, hence $\{3z^2 = y^2\}$ z^2 is a multiple of 3, hence z is a multiple of 3, hence z^2 is a multiple of 9, hence $\{3z^2 = y^2\}$ y^2 is a multiple of 27, hence y is a multiple of 9, etc.. The only natural that is a multiple of 3, 9, 27, .. equals 0.

(End of Appendix.)

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17 March 1994