## An exotic theorem !? (for the record)

One of these days, John Seegers communicated the following theorem: For any relation $r$,

$$[\ \dagger\neg\dagger\neg\dagger r \ \equiv\ \neg\dagger\neg\dagger r\ ]\ ,$$

in which $\dagger p$, the transitive closure of $p$, is defined as the strongest solution of the equation

$$x:\ [\ p \vee p;x \Rightarrow x\ ]\ .$$

The purpose of this note is, besides recording the theorem itself, to present an orderly designed proof, which was concocted by the ETAC on 25 Jan 94. The proof consists in a layered unraveling of the complicated shape of the demonstrandum, while keeping the interfaces between the layers as thin as possible.

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First, we spell out the defining properties of $\dagger p$:

$$\text{(0)}\qquad [\ p \vee p;x \Rightarrow x\ ] \ \Rightarrow\ [\ \dagger p \Rightarrow x\ ] \qquad (\forall x)$$

— the extremity of $\dagger p$ —

$$\text{(1a)}\qquad [\ p \Rightarrow \dagger p\ ]$$

$$\text{(1b)}\qquad [\ p;\dagger p \Rightarrow \dagger p\ ]\ .$$

Second, we slightly generalize the theorem into

(2) for transitive $s$ (i.e. $[s;s \Rightarrow s]$),
$$[\psi \neg t \neg s \equiv \neg t \neg s] .$$

Then, Seeger's theorem follows because $\psi r$ is transitive.

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Now, the first thing we do is to ignore the inner structure of the subexpression $\neg \psi \neg s$ by naming it:
$$[a \equiv \neg \psi \neg s] .$$
Our demonstrandum then becomes $[\psi a \equiv a]$, which we tackle in isolation:

$$
\begin{array}{ll}
 & [\psi a \equiv a] \\
\equiv & \quad \{ [t a \Leftarrow a], \text{ see (1a)} \} \\
 & [\psi a \Rightarrow a] \\
\Leftarrow & \quad \{ \text{extremity of } \psi a, \text{ see (0)} \} \\
 & [a \vee a;a \Rightarrow a] \\
\equiv & \quad \{ \text{pred. calc.} \} \\
 & [a;a \Rightarrow a] .
\end{array}
$$

As a result, proof obligation (2) can be rephrased as

(3) for $a$ and $s$ such that
$[a \equiv \neg \psi \neg s]$ and $s$ is transitive,
$$[a;a \Rightarrow a]$$

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Next, we proceed in very much the same way by ignoring the inner structure of the subexpression $\curlyvee\neg s$ in the expression for $a$. With $b$ such that

$$[b \equiv \curlyvee\neg s] \qquad (\text{i.e. } [b \equiv \neg a]) \ ,$$

our demonstrandum becomes $[\neg b ; \neg b \Rightarrow \neg b]$, which we tackle as follows:

$$[\neg b ; \neg b \Rightarrow \neg b]$$

$\equiv$ { contra-positive, inspired by the observation that $[b \equiv \curlyvee \text{something}]$ }

$$[b \Rightarrow \neg(\neg b ; \neg b)] \ .$$

And here we cannot do much more in the $b$-nomenclature. Thus, proof obligation (3) has been transformed into

(4) for $b$ and $s$ such that $[b \equiv \curlyvee\neg s]$ and $s$ is transitive,

$$[b \Rightarrow \neg(\neg b ; \neg b)]$$

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Finally, we introduce $c$ such that

$$[c \equiv \neg s] \qquad (\text{i.e. } [b \equiv \curlyvee c]) \ ,$$

and our demonstrandum becomes

$$[\curlyvee c \Rightarrow \neg(\neg\curlyvee c ; \neg\curlyvee c)] \ ,$$

which we tackle as follows:

$$[\curlyvee c \Rightarrow \neg(\neg\curlyvee c ; \neg\curlyvee c)]$$

$\Leftarrow$ { extremity of $\curlyvee c$ }

$$[c \vee c;\neg(\neg\Psi c ; \neg\Psi c) \Rightarrow \neg(\neg\Psi c ; \neg\Psi c)]$$

$\equiv$ { pred. calc. }

$$[c \Rightarrow \neg(\neg\Psi c ; \neg\Psi c)]$$
$$\wedge\ [c;\neg(\neg\Psi c ; \neg\Psi c) \Rightarrow \neg(\neg\Psi c ; \neg\Psi c)]$$

Re first conjunct

$$[c \Rightarrow \neg(\neg\Psi c ; \neg\Psi c)]$$

$\equiv$ { contra-positive }

$$[\neg\Psi c ; \neg\Psi c \Rightarrow \neg c]$$

$\Leftarrow$ { $[\neg c \equiv s]$ and $[s \Leftarrow s;s]$, hence $[\neg c \Leftarrow \neg c;\neg c]$ }

$$[\neg\Psi c ; \neg\Psi c \Rightarrow \neg c ; \neg c]$$

$\Leftarrow$ { monotonicity }

$$[\neg\Psi c \Rightarrow \neg c]$$

$\equiv$ { contra-positive }

$$[c \Rightarrow \Psi c]$$

$\equiv$ { $\Psi$ is weakening, see (1a) }

true .

Re second conjunct

$$[c;\neg(\neg\Psi c ; \neg\Psi c) \Rightarrow \neg(\neg\Psi c ; \neg\Psi c)]$$

$\equiv$ { Right Exchange }

$$[\sim c ; (\neg\Psi c ; \neg\Psi c) \Rightarrow \neg\Psi c ; \neg\Psi c]$$

$\equiv$ { ; is associative }

$$[(\sim c ; \neg\Psi c) ; \neg\Psi c \Rightarrow \neg\Psi c ; \neg\Psi c]$$

$\Leftarrow$ { monotonicity }

$$[\sim c ; \neg\Psi c \Rightarrow \neg\Psi c]$$

$\equiv$ { Right Exchange }

$$[c ; \Psi c \Rightarrow \Psi c]$$

$\equiv$ { (1b) }

true .

And this completes our proof

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At the ETAC we used the "confront" operator $!$ defined by

$$[\, x \,!\, y \;\equiv\; \neg(\neg x \,;\, \neg y) \,] \;,$$

but the only thing we did with it is use it to abbreviate an ugly expression like

$$\neg(\neg \Psi c \,;\, \neg \Psi c)$$

as the more friendly looking

$$\Psi c \,!\, \Psi c$$

It has been my (=WF) decision to not use the abbreviation in this note

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I showed the theorem to Jaap van der Woude and he conjectured – from his experience in topology – that Seeger's theorem would probably not be a stand-alone theorem, but one from a whole class.

WHJ Feijen,
Eindhoven,
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