Playing with dagger and star, i.e. with transitive closures

There are many different ways in which one can present the regularity calculus, but in any such presentation the unary operator $*$ – star –, to denote the reflexive transitive closure of a relation, will see the light in a pretty early stage. Over the last years (or decades?) it has become customary to introduce $*r$ as the strongest solution of the equation

(0) $\quad x: [J \vee r;x \Rightarrow x]$.

But, in fact, this is a little bit unfortunate because it can also be introduced as the strongest solution of

(1) $\quad x: [J \vee x;r \Rightarrow x]$.

The choice between (0) and (1) being immaterial, we have to make an immaterial choice in an early stage of such a development of the regularity calculus, and that is not very satisfactory.

Instead of starting from the skew equations (0) or (1), it is much nicer to preserve symmetry and to introduce $*r$ as the strongest solution of

(2) $\quad x: [J \vee r \vee x;x \Rightarrow x]$.

From this definition it is, for instance, immediately clear that any solution x satisfies

$$[\,x;x \Rightarrow x\,]$$.

i.e. any solution is transitive, in particular $*r$. The only little disadvantage of (2) over (1) or (0) is that the defining equation for $*r$ has become slightly more complicated. Therefore, Edsger W. Dijkstra proposed to start our investigations from $\dagger$ – dagger –, to denote the ordinary transitive closure. His proposal is to define $\dagger r$ as the strongest solution of

(3) $\quad x:\ [\,r \vee x;x \Rightarrow x\,]$,

and then $*r$ by, for instance,

(4) $\quad [\,*r \equiv \dagger(J \vee r)\,]$.

During one-and-a-half of its session, the ETAC has explored this proposal (in the presence of EWD), and the purpose of this note is to record our findings

† † †

Dagger all by itself

The first thing to be done is to spell out the defining properties of $\dagger r$:

(5a) $\quad [\,r \Rightarrow x\,] \wedge [\,x;x \Rightarrow x\,] \;\Rightarrow\; [\,\dagger r \Rightarrow x\,] \qquad (\forall x)$

– $\dagger r$'s extremity –

(5b) $\quad [\,r \vee \dagger r;\dagger r \Rightarrow \dagger r\,]$

Remark By the theorem of Knaster-Tarski we can strengthen implication (5b) into an

equivalence, but we refrain from doing so because we can travel a long long way without having to appeal to Knaster & Tarski.
(End of Remark.)

Next we do justice to the name that has been attached to tr, viz. that tr is transitive and that t is a closure.

The transitivity of tr, i.e.

(6) $[tr ; tr \Rightarrow tr]$,

immediately follows from (5b).

For t to be a closure we have to prove that it is weakening, monotonic, and idempotent.

It being weakening, i.e.

(7) $[r \Rightarrow tr]$,

immediately follows from (5b).

It being monotonic, i.e.

(8) $[r \Rightarrow s] \Rightarrow [tr \Rightarrow ts]$,

follows from the fact that (3) – the defining equation for tr – has the shape $x : [f.r.x \Rightarrow x]$, with f monotonic in r.

Remark Here we have appealed to the very general theorem that for f monotonic in its first argument and for g.r the strongest solution of $x : [f.r.x \Rightarrow x]$, g is monotonic as well.
(End of Remark.)

The idempotence of t, i.e.

(9) $[\, ttr \equiv tr \,]$,

is an immediate consequence of Lemma (10), with $s := tr$:

(10) for transitive s , $[\, ts \equiv s \,]$

<u>Proof</u> of (10)

Pong: $[\, ts \Leftarrow s \,]$ is okay since t is weakening

Ping: $[\, ts \Rightarrow s \,]$

$\Leftarrow$ { ts's extremity, see (5a) }

$[\, s \Rightarrow s \,] \wedge [\, s;s \Rightarrow s \,]$

$\equiv$ { pred. calc. }

$[\, s;s \Rightarrow s \,]$

$\equiv$ { s is transitive }

true .

(<u>End</u> of Proof.)

This concludes our demonstration that t is a closure.

† † †

There are a number of special relations within our calculus, to wit the constants true, false, and J, and the left-, right-, and middleconditions — the latter also called monotypes —. They are all transitive, and therefore — by lemma (10) — we conclude

(11) $[\, t\,true \equiv true \,]$ $[\, t\,false \equiv false \,]$

(12) $[\, tJ \equiv J \,]$

(13) for r a left- or right-condition, $[\, tr \equiv r \,]$

(14) $[r \Rightarrow J] \Rightarrow [\dagger r \equiv r]$.

In what follows, we will not have much use for (11) and (13), but we do for (12) and (14)

Remark For the proofs of the transitivity of these special relations, the only properties of ; (semi) that we need are that false is a zero-element, J is the identity-element of ;, and that ; is monotonic in either argument. In particular we want to point out that the only junctivity property of ; used so far, is its monotonicity
(End of Remark.)

† † †

Next, we investigate how † behaves towards disjunctions. Well, it does not behave well, except when middle-conditions are involved:

(15) $[r \Rightarrow J] \Rightarrow [\ \dagger(r \vee s) \equiv \dagger r \vee \dagger s\ ]$

Proof [LHS ⇐ RHS] is okay by monotonicity of †. [LHS ⇒ RHS] follows from

$[\dagger(r \vee s) \Rightarrow \dagger r \vee \dagger s]$

⇐ { extremity of $\dagger(r \vee s)$, see (5a) }

$[r \vee s \Rightarrow \dagger r \vee \dagger s]$

$\wedge\ [\ (\dagger r \vee \dagger s) ; (\dagger r \vee \dagger s) \Rightarrow \dagger r \vee \dagger s]$

≡ { † is weakening, on the first conjunct
; over ∨ on the second conjunct }

$[\ \dagger r ; \dagger r \ \vee\ \dagger r ; \dagger s \ \vee\ \dagger s ; \dagger r \ \vee\ \dagger s ; \dagger s$

$\Rightarrow\ \dagger r \vee \dagger s\ ]$

≡ { $[\dagger r ; \dagger r \Rightarrow \dagger r]$ and $[\dagger s ; \dagger s \Rightarrow \dagger s]$ }

$$[tr ; ts \ \vee\ ts ; tr \ \Rightarrow\ tr \vee ts]$$
$\Leftarrow$ { $[r \Rightarrow J]$ from the antecedent of (15), hence $[tr \Rightarrow J]$ from (14) }
$$[J ; ts \ \vee\ ts ; J \ \Rightarrow\ tr \vee ts]$$
$\equiv$ { rel. calc. }
$$[ts \Rightarrow tr \vee ts]$$
$\equiv$ { pred. calc. }
true .

(End of Proof.)

By (15) with $r, s := J, r$, and by (12), we have as a corollary

(16) $\quad [\ t(J \vee r) \ \equiv\ J \vee tr\ ]$,

which will be used shortly. Also we shall use the "lifted" version of (16), which is

(17) $\quad t \circ (J\vee) \ =\ (J\vee) \circ t$,

and which expresses quite clearly that functions $(J\vee)$ and t commute.

Remark We want to point out that up to this point the only junctivity property of ; that we have used is its finite disjunctivity. (End of Remark.)

So much for t all by itself.

* * *

Star is coming up

Now, following Dijkstra's proposal, we let * see the light by defining $*r$ by

(18a) $[*r \equiv \dagger(J \vee r)]$.

Thanks to (16) we now also have

(18b) $[*r \equiv J \vee \dagger r]$,

and in exploring properties of $*$ we now can freely choose definition (18a) or (18b), whichever comes in most handy. We also continue our flirt with lifting, and we lift (18) towards

(19a) $* = \dagger \circ (J \vee)$

(19b) $* = (J \vee) \circ \dagger$.

For doing justice to the usual name attached to $*$, we have to prove that $*r$ is reflexive and transitive, and that $*$ is a closure

The reflexivity of $*r$, i.e.

(20) $[J \Rightarrow *r]$,

immediately follows from (18b).

The transitivity of $*r$, i.e.

(21) $[*r ; *r \Rightarrow *r]$,

is an immediate consequence of the transitivity of $\dagger$, thus:

$*r ; *r$

$\equiv$ { (18a) }

$\dagger(J \vee r) ; \dagger(J \vee r)$

$\Rightarrow$ { transitivity of $\dagger(J \vee r)$ }

$\dagger(J \vee r)$

$\equiv$ { (18a) }

$*r$.

For $*$ to be a closure, we have to demonstrate that it is weakening, monotonic, and idempotent. For this purpose we shall use one of the lifted formulae (19).

Because $\dagger$ and $(J\vee)$ are both weakening and monotonic, so is their functional composition, and, hence, so is $*$.

Because $\dagger$ and $(J\vee)$ are each idempotent, and because they commute — see (17) — , their functional composition is idempotent as well:

$$
\begin{array}{ll}
 & * \circ * \\
= & \quad \{ (19a) \} \\
 & \dagger \circ (J\vee) \circ \dagger \circ (J\vee) \\
= & \quad \{ (17) \text{ on the two middle terms} \} \\
 & \dagger \circ \dagger \circ (J\vee) \circ (J\vee) \\
= & \quad \{ \text{idempotence of } \dagger \text{ and } (J\vee) \} \\
 & \dagger \circ (J\vee) \\
= & \quad \{ (19a) \} \\
 & * \quad .
\end{array}
$$

In summary.

(22) $[\, r \Rightarrow *r \,]$

(23) $[\, r \Rightarrow s \,] \Rightarrow [\, *r \Rightarrow *s \,]$

(24) $[\, **r \equiv *r \,]$ or $* \circ * = *$

* * *

In retrospection, the idempotence of $*$ can be dealt with in very much the same way we dealt with the idempotence of $\dagger$, viz. it is a direct consequence of Lemma (25):

(25) for reflexive and transitive s, $[*s \equiv s]$

<u>Proof</u> of (25)

$*s$
$\equiv$ { (18b) }
$J \vee \dagger s$
$\equiv$ { s is transitive, hence $[\dagger s \equiv s]$ }
$J \vee s$
$\equiv$ { s is reflexive, i.e. $[J \Rightarrow s]$ }
s

(<u>End</u> of Proof.)

Finally, we mention without proof

(26) $[*\text{false} \equiv J]$ $[*\text{true} \equiv \text{true}]$

(27) $[*J \equiv J]$

(28) $[r \Rightarrow J] \Rightarrow [*r \equiv J]$

* * *

The transitivity of $*r$ — see (21) — is subsumed in the stronger

(29) $[*r ; *r \equiv *r]$.

<u>Proof</u>

$*r ; *r$
$\equiv$ { (18b) }
$(J \vee \dagger r) ; (J \vee \dagger r)$
$\equiv$ { ; over $\vee$ }{ rel. calc. }

$J \vee \dagger r \vee \dagger r ; \dagger r$

$\equiv$ { pred. calc. using $[\dagger r ; \dagger r \Rightarrow \dagger r]$ }

$J \vee \dagger r$

$\equiv$ { (18b) }

$\star r$.

(End of Proof.)

Remark Still, we have not used more of ; than its finite disjunctivity.
(End of Remark.)

Dagger and Star together

Here we examine how $\dagger$ and $\star$ act among each other. We find

(30a) $[\dagger \star r \equiv \star r]$ or $\dagger \circ \star = \star$

(30b) $[\star \dagger r \equiv \star r]$ or $\star \circ \dagger = \star$

Proofs

$\dagger \circ \star$

$=$ { (19a) }

$\dagger \circ \dagger \circ (J\vee)$

$=$ { $\dagger$ is idempotent }

$\dagger \circ (J\vee)$

$=$ { (19a) }

$\star$

$\star \circ \dagger$

$=$ { (19b) }

$(J\vee) \circ \dagger \circ \dagger$

$=$ { $\dagger$ is idempotent }

$(J\vee) \circ \dagger$

$=$ { (19b) }

$\star$

(End of Proofs.)

Next, in combination with ; we find

(31a) $[\ast r ; \dagger r \;\equiv\; \dagger r]$

(31b) $[\dagger r ; \ast r \;\equiv\; \dagger r]$

Proof of (31a)

$\ast r ; \dagger r$
$\equiv$ { (18b) }
$(J \vee \dagger r) ; \dagger r$
$\equiv$ { ; over $\vee$ }{ rel. calc. }
$\dagger r \vee \dagger r ; \dagger r$
$\equiv$ { $\dagger r$ is transitive }
$\dagger r$

(End of Proof.)

Sad Intermezzo

The above was written "aus einem Guß" one day early January 1994. It is early October now. The reason for that enormous delay is that fun is over now. In what follows, we wish to relate the symmetric and the skew equations for dagger and star, but the ensuing proofs are far from nice or crisp.