Courtesy AJM van Gasteren — and Fusion Lemmata —

We consider equation

$$x: [J \vee x;a \equiv x] ,$$

which, by definition, has $*a$ as its strongest solution.

We also consider equation

$$x: [b \vee x;a \equiv x] ,$$

which, by a well-known theorem, has $b;*a$ as its strongest solution. The proofs of this well-known theorem have been cumbersome for quite a while. Over the last few years or so the proofs have been cleaned up by the employment of "factors" or similar devices, but still remained unsatisfactory. These times are over now, thanks to the emergence of so-called Fusion Lemmata (originating from Ed Voermans ?). One such lemma is — for notation, see WF168 —

Fusion Lemma

$$\mu h = f.\mu g$$
$$\Leftarrow$$
$$h \circ f = f \circ g \quad \wedge \quad h \text{ monotonic}$$
$$\wedge \quad f \text{ is a lower adjoint}$$

(**End** of Fusion Lemma.)

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We will use the Lemma to derive a function h such that equation $x : [h.x \equiv x]$ has $b ; *a$ as its strongest solution, i.e. such that $[\mu h \equiv b ; *a]$. In view of the consequent of the Fusion Lemma we can conclude $[\mu h \equiv b ; *a]$ whenever we choose f and g such that

(0) $[f.x \equiv b ; x]$

$[\mu g \equiv *a]$

The requirement on g is met whenever

(1) $[g.x \equiv J \vee x ; a]$

Now we derive h by the remaining obligation to show that, for choices (0) and (1) for f and g, the antecedent of the Fusion Lemma is met.

- f is a lower adjoint: this is okay since the f given by (0) is universally disjunctive
- for any x,

$$
\begin{array}{ll}
 & [(h \circ f).x \equiv (f \circ g).x] \\
= & \quad \{\text{definitions of } \circ,\ f, \text{ and } g\} \\
 & [h.(b ; x) \equiv b ; (J \vee x ; a)] \\
= & \quad \{\text{rel. calc.}\} \\
 & [h.(b ; x) \equiv b \vee b ; x ; a] \\
\Leftarrow & \quad \{\text{instantiation}\} \\
 & (\forall y :: [h.y \equiv b \vee y ; a]),
\end{array}
$$

and this last line determines our choice for h !

- Function h, chosen above, is monotonic indeed.

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It was Netty van Gasteren who, after having read WF168, observed the above possibility for proving that $b;*a$ is the strongest solution of $x: [b \vee x;a \equiv x]$. Her proof beats all previous ones. Also: one up for Fusion Lemmata that act as logical firewalls against the laborious ping-pong (or: solves - extremity) arguments.

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