Jaap van der Woude's reply to WF167

(This note contains nothing new and nothing in it is mine. It is written for my own files, mainly.)

In WF167, I proved the following theorem:

For ∞s the weakest solution of the "homogeneous" equation

(0) $\quad x: [\, s;x \equiv x \,]$, and

for r any "particular" solution of the "inhomogeneous" equation

(1) $\quad x: [\, t \vee s;x \equiv x \,]$,

we have that

$r \vee \infty s$ is the weakest solution of the "inhomogeneous" equation (1).

* * *

When Jaap van der Woude saw the above theorem he immediately recognized it as a very special instance of the following —notation to be explained shortly— very general

Fusion Lemma

$$\nu h = f.\nu g$$
$$\Leftarrow$$
$$h \circ f = f \circ g \quad \wedge \quad f \text{ is an upper adjoint}$$

(*End* of Fusion Lemma.)

Notation "νh" is the classical notation for the greatest solution of equation x: $x = h.x$ (or — Knaster/Tarski — x: $x \leq h.x$). In case h is a predicate transformer νh is the weakest solution of x: $[x \equiv h.x]$ (or x: $[x \Rightarrow h.x]$).

The notion "upper adjoint" is related to Galois connexions. Function f is an upper adjoint means that there exist a function f^b such that $[f^b.x \leq y \;\equiv\; x \leq f.y]$. One of the theorems of this field — not to be proven here — is that a universally conjunctive predicate transformer is an upper adjoint.

So much for this jargon.

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Now, with h, g, and f defined by

$$[h.x \;\equiv\; t \vee s;x]$$

$$[g.x \;\equiv\; s;x]$$

$$[f.x \;\equiv\; r \vee x]$$

WF167's theorem can now be rendered as

$$\nu h = f.\nu g \quad ,$$

which is the consequent of the Fusion Lemma. So, in order to prove the theorem we only need to check the Lemma's antecedent for our specific functions f, g, and h. Here we go.

As for f being an upper adjoint, it suffices to observe that f is universally conjunctive, which it is.

As for $h \circ f = f \circ g$, we observe that for any x

$$
\begin{array}{cl}
 & h.(f.x) \\
= & \{\text{ definitions of } h \text{ and } f \} \\
 & t \vee s;(r \vee x) \\
= & \{\text{ rel. calc } \} \\
 & t \vee s;r \ \vee \ s;x \\
= & \{\ r \text{ is a particular solution of (1) } \} \\
 & r \vee s;x \\
= & \{\text{ definitions of } f \text{ and } g \} \\
 & f.(g.x) \quad .
\end{array}
$$

End of Van der Woude's proof of WF167's theorem! I think it is beautifully short, of course at the expense of the Fusion Lemma. But I think the Fusion Lemma is beautiful too, and valuable to have. It has eliminated a ping-pong argument (or rather: a "solves" and "extremity" argument). It is one of a body of theorems to smoothen reasoning about fixpoints, which is a laudable goal. (J.L.A. van de Snepscheut's μ-calculus aims at smoothening reasoning about extreme solutions in general, which is even more ambitious, but — as it stands — at the price of more baroque calculus.)

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Now, for the sake of completeness, or just for my files, or even just for fun we supply a proof of the Fusion Lemma.

<u>Proof</u> Ping - Pong

- $f.rg \le rh$
  $\Leftarrow$ { extremity of rh ; Knaster - Tarski ; remember equation $x:\ x \le h.x$ }
  $f.rg \le h.(f.rg)$
  $\Leftarrow$ { antecedent $h \circ f = f \circ g$, in particular the part $f \circ g \le h \circ f$ }
  $f.rg \le f.(g.rg)$
  $=$ { rg solves $x:\ x = g.x$ }
  $f.rg \le f.rg$
  $=$ {}
  true .

- $rh \le f.rg$
  $=$ { f is an upper adjoint }
  $f^{b}.rh \le rg$
  $\Leftarrow$ { extremity of rg ; Knaster - Tarski ; remember equation $x:\ x \le g.x$ }
  $f^{b}.rh \le g.(f^{b}.rh)$
  $=$ { f is an upper adjoint }
  $rh \le f.(g.(f^{b}.rh))$
  $\Leftarrow$ { antecedent $h \circ f = f \circ g$, in particular the part $h \circ f \le f \circ g$ }
  $rh \le h.(f.(f^{b}.rh))$
  $=$ { rh solves $x:\ x = h.x$ }
  $h.rh \le h.(f.(f^{b}.rh))$
  $\Leftarrow$ { h monotonic : a forgotten premise of the Fusion Lemma }
  $rh \le f.(f^{b}.rh)$
  $=$ { f is an upper adjoint }
  $f^{b}.rh \le f^{b}.rh$
  $=$ {}
  true .

(End of Proof.)

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