# One down for the relational calculus

(whether you like it or not)

The following problem was communicated by Richard S. Bird. For any binary relation R and subset S on some universe, set Min.R.S is defined by, for all x

$$
\begin{aligned}
& x \in \text{Min}.R.S \\
& \equiv \\
& x \in S \;\wedge\; (\forall y : y \in S : \; x R y) \;.
\end{aligned}
$$

The problem is to prove the

<u>Theorem</u>

For any two preorders X and Y there exists a preorder Z such that for all S

$$\text{Min}.Z.S = \text{Min}.Y.(\text{Min}.X.S) \;.$$

(A preorder is a reflexive and transitive binary relation.)

(<u>End</u> of Theorem.)

More specifically, Bird's assignment was to prove the theorem using the predicate calculus.

* * *

The very first thing we (=WF) did to the problem was change it into an exercise in the relational calculus rather than the predicate calculus. After all, Bird's theorem was a theorem on relations and we (= WF and many others) had acquired some experience in relational calculation recently. So here we had a good opportunity for experimentation. Meanwhile these experiments have been done and some of them have been reported — e.g. [HD], [WF147], [JvdW] —, and the outcome should be sobering, to put it mildly. Now the time has come to return to Richard Bird's original assignment, and see what the ordinary predicate calculus can do for us.

* * *

The following proof is at best a minor variation of the proofs Lex Bijlsma, Anne Kaldewaij, and Jaap van der Woude have designed earlier — and independently —. As far as notation is concerned we shall stay closest to Kaldewaij's. (It need not amaze us that the three gentlemen have designed the "same" proof, since —as we shall see— the task is largely of the form "there is only one thing you can reasonably do".)

Our design starts focussing on the equation

$$\text{for all } S: \quad Min.Z.S \;=\; Min.Y.(Min.X.S)$$

that we try to solve for $Z$. To that end we consider a calculation that begins with: for any $u$ and $S$,

$$
\begin{array}{ll}
 & u \in Min.Y.(Min.X.S) \\
= & \quad \{\text{definition of } Min.Y\} \\
 & u \in Min.X.S \;\wedge\; (\forall v: v \in Min.X.S : u\,Y\,v) \\
= & \quad \{\text{definition of } Min.X \text{, twice}\} \\
 & u \in S \;\wedge\; (\forall w: w \in S : u\,X\,w) \\
 & \wedge\; (\forall v: v \in S \wedge (\forall w: w \in S : v\,X\,w) : u\,Y\,v)
\end{array}
$$

$$
\begin{array}{ll}
= & \{\text{trading in the last conjunct}\} \\
(*0) & u \in S \\
(*1) & \wedge\ (\forall w : w \in S : u\, X\, w) \\
(*2) & \wedge\ (\forall v : v \in S : (\forall w : w \in S : v\, X\, w) \Rightarrow u\, Y\, v)\ ,
\end{array}
$$

and that ends with

$$
\begin{array}{ll}
(**) & u \in S\ \wedge\ (\forall w : w \in S : u\, Z\, w) \\
= & \quad \{\text{definition of } Min.Z\} \\
 & u \in Min.Z.S \quad .
\end{array}
$$

The task that remains is to bridge the gap between the expressions $(*)$ and $(**)$. The two universal quantifications $(*1)$ and $(*2)$ are ready to be joined. The only problem is in the term of $(*2)$ which still mentions $S$, in particular subexpression $(\forall w : w \in S : v\, X\, w)$. For that expression we now observe

$$
\begin{array}{ll}
 & (\forall w : w \in S : v\, X\, w) \\
\Rightarrow & \quad \{u \in S,\ \text{i.e. } (*0)\} \\
 & v\, X\, u \\
\Rightarrow & \quad \{\text{predicate calculus}\} \\
 & (\forall w : w \in S : v\, X\, u) \\
= & \quad \{(*1)\} \\
 & (\forall w : w \in S : v\, X\, u\ \wedge\ u\, X\, w) \\
\Rightarrow & \quad \{X \text{ is transitive}\} \\
 & (\forall w : w \in S : v\, X\, w) \quad \text{, i.e. the first line.}
\end{array}
$$

Hence, $(\forall w : w \in S : v\, X\, w) \equiv v\, X\, u$ , and now we can continue our calculation from $(*)$ .

<u>Remark</u> The first step in the above little calculation is not a rabbit at all. The patented way to remove $S$ from the expression $(\forall w: w \in S: v \, X \, w)$ is by a proper instantiation. In the context in which that expression resides there are only two elements of $S$ available, viz. $v$ and $u$. Instantiation with $v$ would yield $v \, X \, v$ which, in view of $X$'s reflexivity, equivales $true$, and therefore would make no sense.
(<u>End</u> of Remark.)

We now have

$$
\begin{array}{ll}
 & (*) \\
= & \quad \{\text{by the above result}\} \\
 & u \in S \\
 & \wedge \; (\forall w: w \in S: u \, X \, w) \\
 & \wedge \; (\forall v: v \in S: v \, X \, u \;\Rightarrow\; u \, Y \, v) \\
= & \quad \{\text{joining the terms}\} \\
 & u \in S \\
 & \wedge \; (\forall w: w \in S: u \, X \, w \\
 & \qquad\qquad \wedge \; (w \, X \, u \;\Rightarrow\; u \, Y \, w)\;) \\
= & \quad \{\text{calculus of relations}\} \\
 & u \in S \\
 & \wedge \; (\forall w: w \in S: u \, (X \wedge (\sim X \Rightarrow Y)) \, w \;) \\
= & \quad \{\text{choose:} \\
 & \qquad [\, Z \equiv X \wedge (\sim X \Rightarrow Y) \,] \quad \} \\
 & (**)
\end{array}
$$

* * *

The second part of our proof of Bird's theorem consists of showing that the result relation $Z$ given by

$$[\ Z \ \equiv \ X \wedge (\sim X \Rightarrow Y)\ ]$$

is a preorder whenever $X$ and $Y$ are preorders. We supply this proof for completeness's sake (and for reasons of comparison with the corresponding proof in the relational calculus), although nothing is fascinating about it — a machine or a freshman could probably do it —.

- $Z$ is reflexive, i.e. $u\ Z\ u$ for all $u$:

$$\begin{array}{ll} & u\ Z\ u \\ = & \quad \{\text{definition of } Z\} \\ & u\ X\ u \ \wedge \ u\ (\sim X \Rightarrow Y)\ u \\ \Leftarrow & \quad \{\text{calculus}\} \\ & u\ X u \ \wedge \ u\ Y u \\ = & \quad \{X \text{ and } Y \text{ are reflexive}\} \\ & \text{true} \ . \end{array}$$

- $Z$ is transitive, i.e for all $u, v, w$

$$u\ Z\ v \ \wedge \ v\ Z\ w \ \Rightarrow \ u\ Z\ w$$

First we spell out the antecedent. It is the conjunction of

(a) $u\ X\ v$ (b) $v\ X\ w$
(c) $u (\sim X) v \Rightarrow u\ Y\ v$ (d) $v\ (\sim X)\ w \Rightarrow v\ Y\ w$

Next we spell out the consequent. It is the conjunction of

(e) $u\ X\ w$ (f) $u\ (\sim X)\ w \Rightarrow u\ Y\ w$ .

Now we observe for (e)

$$
\begin{array}{ll}
 & u\, X\, w \\
\Leftarrow & \quad \{ X \text{ is transitive} \} \\
 & u\, X\, v \;\wedge\; v\, X\, w \\
= & \quad \{ \text{(a) and (b)} \} \\
 & true \quad ,
\end{array}
$$

and for (f)

$$
\begin{array}{ll}
 & u\, Y\, w \\
\Leftarrow & \quad \{ Y \text{ is transitive} \} \\
 & u\, Y\, v \;\wedge\; v\, Y\, w \\
\Leftarrow & \quad \{ \text{(c) and (d)} \} \\
 & u\, (\sim X)\, v \;\wedge\; v\, (\sim X)\, w \\
= & \quad \{ \text{definition of } \sim \} \\
 & v\, X\, u \;\wedge\; w\, X\, v \\
\Leftarrow & \quad \{ X \text{ is transitive} \} \\
 & w\, X\, u \\
= & \quad \{ \text{definition of } \sim \} \\
 & u\, (\sim X)\, w \quad , \text{ i.e. the antecedent of (f).}
\end{array}
$$

* * *

This is it. Is there a moral? Perhaps, but in the first place there is the observation that a proof of Bird's theorem is far more easily constructed when using the predicate calculus than in case of using the relational calculus. This is a surprising outcome since one would guess that the relational calculus were the symbolism par excellence to tackle such a theorem on relations. Quod non.

We cannot leave it at this because we would like to have a technical explanation for the observed phenomenon. One possibility is this. At some point we derived a very crucial equivalence, viz.

$$(***) \qquad (\forall w : w \in S : v \, X \, w) \equiv v \, X \, u \ .$$

It provided the link between the head and the tail of our main calculation. And it is precisely here, in expressing (xxx), where the relational calculus falls down. In a "dummy-free" notation like the relational calculus an expression like $v \, X \, u$ simply cannot see the light, so that nothing else can be done than to "program around it".

It can be argued that a formalism with reduced manipulative possibilities has the advantage that it reduces the solution spaces in solving problems. But at the same time we should then be willing to accept that the reduction may be so dramatic that solution spaces can collapse to empty. Here we wish to transmit to the relational calculators a question posed by Carel S. Scholten. Give five relations PQ, QR, RS, PR, and QS, find a relational expression Z such that

$$\begin{array}{l} p \, Z \, s \\ \equiv \\ (\exists q, r :: p \, PQ \, q \ \wedge \ q \, QR \, r \ \wedge \ r \, RS \, s \\ \qquad\qquad \wedge \ p \, PR \, r \ \wedge \ q \, QS \, s \,) \ . \end{array}$$

It looks as if the straightjacket of the relational calculus, if not for theoretical reasons then for practical reasons anyway, cannot be maintained.

W.H.J. Feijen,
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Eindhoven

[HD] Richard Bird's Problem,
Technical Report by Henk Doornbos

[WF147] Designing a proof for R.S. Bird's theorem on pre-orders,
Technical Report by W.H.J. Feijen

[JvdW] Free style spec wrestling II: preorders,
Technical Report by Jaap van der Woude
(to be published in The Squigollist.)