Designing a proof for R.S. Bird's theorem on pre-orders.

We are given some fixed, anonymous universe of things. For any binary relation R on that universe and for any subset S on that same universe subset $Min.R.S$ is defined as follows: for any x,

$$x \in Min.R.S$$
$$\equiv$$
$$x \in S \wedge (\forall y :: y \in S \Rightarrow x R y) .$$

The theorem to be proven is that for any two pre-orders X and Y, there exists a pre-order Z satisfying

$$Min.Z = Min.Y \circ Min.X ,$$

or -equivalently-, for all S

(0) $\quad Min.Z.S = Min.Y.(Min.X.S)$.

(A pre-order is a binary relation that is both reflexive and transitive.)

* * *

The above was communicated to us by Richard S. Bird as an exercise in the predicate calculus. It was accompanied by the warning that proving the theorem had been "surprisingly difficult". In view of our recent involvement in the relational calculus and in view of the announced difficulty of the exercise, Bird's theorem came -more or lessas a gift from heaven, because now there was an opportunity to put the relational calculus at work for solving a nontrivial problem. Therefore we took the liberty of changing Bird's exercise into an exercise in the relational calculus, with the purpose of testing the latter's potential.

However, it turned out that, more than anything else, the exercise became an exercise in proof development. In our first effort to prove the theorem we were insufficiently aware of this, and in finding a proof we would proceed so uncautiously that, in the end, we got completely stuck. Warned and alarmed by such a miserable performance, we started afresh, this time obeying all our current rules for proof construction. The ensuing result was a proof of Bird's theorem with almost every step pre-ordained, thus offering no surprises at all. The main purpose of this note has now become to exhibit that development.

* * *

This text will not be self-contained in that it uses the relational calculus. The appeal to that calculus will, however, be quite modest and not go beyond e.g. AvG92/WF140: "An introduction into the relational calculus". For completeness's sake we supply an appendix mentioning the most important calculational rules employed in this note.

* * *

Translating the problem

The problem, as stated, is formulated in terms of subsets of and binary relations on a given universe. Since we wish to tackle the problem using the relational calculus we first translate sets into binary relations.

There are two standard ways of pairing sets and relations. One is to associate sets with relations called monotypes, and the other is to couple them to relations called leftconditions. Because there is a one-to-one correspondence between monotypes and leftconditions, the choice is irrelevant from a mathematical point of view. However, we choose to represent sets by leftconditions because it so happens that with that choice the ensuing formulae become simpler by almost one order of magnitude.

By convention, set S and leftcondition S — no confusion will arise from overloading name S — will be coupled by the rule

$$(\forall x,z :: \; x \in S \;\equiv\; x\, S\, z) \; .$$

(The fact that the binary relation S defined by this rule indeed matches the notion of S being a leftcondition as we know it from the relational calculus, is not demonstrated here.)

Theorem (0) is entirely expressed in expressions of the form $Min.Z.S$. For set & leftcondition S, we therefore seek to translate set $Min.Z.S$ into a relation that is a leftcondition. We propose that

$$Min.Z.S = A$$

where A is the largest set satisfying – see definition of Min –

$$(\forall x :: x \in A \Rightarrow x \in S \wedge (\forall y :: y \in S \Rightarrow x\, Z\, y)) \;,$$

or – equivalently –

(i) $(\forall x :: x \in A \Rightarrow x \in S)$, and

(ii) $(\forall x :: x \in A \Rightarrow (\forall y :: y \in S \Rightarrow x\, Z\, y))$.

First we translate (i) into the relational format:

(i)
= { introduction of an additional dummy }
$(\forall x,z :: x \in A \Rightarrow x \in S)$
= { using the coupling rule for S }
$(\forall x,z :: x \in A \Rightarrow x\, S\, z)$
= { on the premise that also A satisfies the coupling rule, i.e. relation A is a leftcondition }
$(\forall x,z :: x\, A\, z \Rightarrow x\, S\, z)$
= { definition of [] }
$[A \Rightarrow S]$.

Next we observe

(ii)
= { pred. calc. }
$(\forall x,y :: x \in A \wedge y \in S \Rightarrow x\, Z\, y)$

$=$ { additional dummy }
$(\forall x,y :: (\exists z :: x \in A \wedge y \in S) \Rightarrow x Z y)$
$=$ { coupling rule: given for S and demanded for A }
$(\forall x,y :: (\exists z :: x A z \wedge y S z) \Rightarrow x Z y)$
$=$ { definition of $\sim$ }
$(\forall x,y :: (\exists z :: x A z \wedge z (\sim S) y) \Rightarrow x Z y)$
$=$ { definition of $;$ }
$(\forall x,y :: x (A ; \sim S) y \Rightarrow x Z y)$
$=$ { definition of [] }
$[A ; \sim S \Rightarrow Z]$.

Summarizing, we have that set Min.Z.S translates into the weakest relation A satisfying

(*) $[A \Rightarrow S] \wedge [A ; \sim S \Rightarrow Z]$ and

(**) A is a leftcondition .

At this point we have a stroke of good luck, since the weakest A satisfying just (*) is a leftcondition whenever S is — shown below — so that demand (**) of A is for free. For future use we now completely spell out the fact that A is the weakest relation satisfying (*) :

(a0) $[A \Rightarrow S]$

(a1) $[A ; \sim S \Rightarrow Z]$

(a2) $[W \Rightarrow S] \wedge [W ; \sim S \Rightarrow Z] \Rightarrow [W \Rightarrow A]$ $(\forall W)$

Now we show that (**) follows from properties (a) and from S being a leftcondition.

<u>Proof</u> We show that A is a leftcondition by using the relational definition $[A;\text{true} \Rightarrow A]$ rather than a pointwise definition of leftconditions.

$$
\begin{array}{ll}
 & [A;\text{true} \Rightarrow A] \\
\Leftarrow & \quad \{ \text{(a2) with } W := A;\text{true} \} \\
 & [A;\text{true} \Rightarrow S] \wedge [A;\text{true};\sim S \Rightarrow Z] \\
= & \quad \{ S \text{ is a leftcondition and} \\
 & \quad\ \ \sim S \text{ is a rightcondition} \} \\
 & [A;\text{true} \Rightarrow S;\text{true}] \wedge [A;\sim S \Rightarrow Z] \\
\Leftarrow & \quad \{ \text{monotonicity of } ; \} \quad \{ \text{(a1)} \} \\
 & [A \Rightarrow S] \\
= & \quad \{ \text{(a0)} \} \\
 & \text{true} .
\end{array}
$$

(<u>End</u> of Proof.)

* * *

Now we are ready to translate our target expression

(0) $\quad Min.Z.S = Min.Y.(Min.X.S)$.

With $\quad A = Min.Z.S$
$\quad\quad\quad B = Min.Y.C$
$\quad\quad\quad C = Min.X.S$,

(0) can be rewritten as

$$A = B \ .$$

With our representation of sets by leftconditions, (0) can be rewritten as

$$[A \equiv B] \ ,$$

with as givens that S, A, B, and C are leftconditions satisfying properties (a), (b), and (c) — the latter two to be displayed in a moment — .

Bird's theorem is that for preorders X and Y there exists a preorder Z such that $[A \equiv B]$ holds. For the definition of X being a preorder we will use the relational definitions

$[J \Rightarrow X]$ for the reflexivity of X,

and $[X;X \Rightarrow X]$ for the transitivity of X.

* * *

Before we embark on the construction of a proof we first tabulate all the givens. Our advice to the reader is to physically isolate this table from the rest of the text — for instance by Xeroxing it — and to keep it ready for inspection all through the process of proof construction. Also he should use it to keep track of which givens have been used in the proof "so far", because at some point during the proof construction this record plays an important heuristical rôle.

Here is the table.

The table of givens

(a0) $[A \Rightarrow S]$

(a1) $[A ; \sim S \Rightarrow Z]$

(a2) $[W \Rightarrow S] \wedge [W ; \sim S \Rightarrow Z] \Rightarrow [W \Rightarrow A]$

(b0) $[B \Rightarrow C]$

(b1) $[B ; \sim C \Rightarrow Y]$

(b2) $[W \Rightarrow C] \wedge [W ; \sim C \Rightarrow Y] \Rightarrow [W \Rightarrow B]$

(c0) $[C \Rightarrow S]$

(c1) $[C ; \sim S \Rightarrow X]$

(c2) $[W \Rightarrow S] \wedge [W ; \sim S \Rightarrow X] \Rightarrow [W \Rightarrow C]$

S, A, B, and C are leftconditions

X and Y are preorders, i.e.

$[J \Rightarrow X]$ $\qquad$ $[J \Rightarrow Y]$

$[X ; X \Rightarrow X]$ $\qquad$ $[Y ; Y \Rightarrow Y]$

Constructing a proof

Our proof will be entirely calculational. For just checking the correctness of the calculations the hints suffice. But we want to do more. For a number of crucial steps we wish to explain *why* we did those steps. Therefore we will annotate our calculations with "reasons" containing heuristical considerations. These heuristical considerations are important to the extent that they may reveal that certain steps that look like rabbits are, upon closer scrutiny, not rabbits at all. In fact, they may reveal that our forthcoming proof is largely pre-ordained.

* * *

We have to prove the existence of a relation Z such that

$$[A \equiv B] \quad \wedge \quad Z \text{ is a preorder} \ .$$

We do this by constructing at least one witness. Because the second conjunct is too general a requirement, we start focussing on the first one. Because all our givens about A and B occur in implications, we rewrite $[A \equiv B]$ as the conjunction of $[A \Rightarrow B]$ and $[B \Rightarrow A]$, and we tackle these conjuncts separately.

Re $[A \Rightarrow B]$

$$\begin{array}{ll} & [A \Rightarrow B] \\ \Leftarrow & \qquad \{ \text{(b2) with } W := A \} \end{array}$$

$$[A \Rightarrow C] \wedge [A ; \sim C \Rightarrow Y]$$

<u>Reason</u> Expression $[A \Rightarrow B]$ has A in the antecedent and B in the consequent. The only rule that can handle a B in the consequent is rule (b2) and the only rule that can handle an A in the antecedent is rule (a0). The reason to reject an application of (a0) is that it would genuinely strengthen $[A \Rightarrow B]$, whereas the application of (b2) is, in fact, an equivalence preserving step: the whole bunch of formulae (b0), (b1), and (b2) can be rewritten as the single and equivalent

$$(*) \quad [W \Rightarrow C] \wedge [W ; \sim C \Rightarrow Y] \equiv [W \Rightarrow B] \qquad (\forall W)$$

– as the reader may verify – . In case of choice, equivalence preserving steps are always to be preferred over other ones, that is to say: there should be very good reasons for neglecting this heuristic rule.

In the meantime, the reader may wonder why in the first place we tabulated the expanded formulae (b) instead of the much more compact (*). The argument for this is that (*) is too entangled. It acts as a container and hampers direct access to the properties (b0) and (b1). The reason why we rendered (b2) as an implication rather than an equivalence is that, without loss of mathematical content, it reduces our manipulative possibilities and thereby the search-space in which our ultimate proof is to be found.
(<u>End</u> of Reason.)

Next we tackle the conjuncts $[A \Rightarrow C]$ and $[A ; \sim C \Rightarrow Y]$ separately.

$$
\begin{array}{ll}
\bullet & [A \Rightarrow C] \\
\Leftarrow & \quad \{ (c2) \text{ with } W := A \text{ , heuristics as before} \} \\
 & [A \Rightarrow S] \wedge [A ; \sim S \Rightarrow X] \\
= & \quad \{ (a0) \text{ for the first conjunct} \} \\
 & [A ; \sim S \Rightarrow X] \\
\Leftarrow & \quad \{ (a1) \} \\
 & [Z \Rightarrow X] \text{ .}
\end{array}
$$

<u>Reason</u> The expression $[A ; \sim S \Rightarrow X]$ could also have been strengthened by weakening A via (a0) or by strengthening X via (c1). However, we have to bear in mind that the theorem should hold for <u>all</u> S, and that sooner or later we have to remove all occurrences of S - and, in its wake, all occurrences of A, B, and C - from our demonstrandum. The rules (a1), (b1), and (c1) are the only ones that can do this for us.
(<u>End</u> of Reason.)

Meanwhile we have encountered the first constraint to be imposed on Z, viz.

(1) $\qquad [Z \Rightarrow X]$.

$$
\begin{array}{ll}
\bullet & [A ; \sim C \Rightarrow Y] \\
= & \quad \{ \text{pred. calc.} \} \\
 & [A ; \sim C \wedge A ; \sim C \Rightarrow Y] \\
\Leftarrow & \quad \{ (a0) \text{, and } (c0) \text{ in the} \\
 & \qquad \text{form } [\sim C \Rightarrow \sim S] \quad \} \\
 & [S ; \sim C \wedge A ; \sim S \Rightarrow Y]
\end{array}
$$

<u>Reason</u> The table of givens provides three possibilities for strengthening $[A;\sim C \Rightarrow Y]$. One of them strengthens consequent Y via (b1), but this removes Y from the expression which is not to be recommended. The two other possibilities weaken the antecedent by weakening A via (a0) and $\sim C$ via (c0). Which one do we choose? Here we follow a rule that we owe to Edsger W. Dijkstra, and apply both possibilities simultaneously while seeing to it that the resulting expression is as weak as possible. It is this latter goal that explains the emergence of the $\wedge$ in the antecedent of the newly formed expression.

At this point it is nice to add that, had we allowed ourselves a coarser strengthening than the one resulting from Dijkstra's rule, the rest of the proof would have come to an unsuccessful end. Here we may have an explanation for Richard Bird's characterization of the problem: "surprisingly difficult".
(<u>End</u> of Reason.)

We continue our calculation:

$$
\begin{array}{ll}
 & [S;\sim C \ \wedge\ A;\sim S \ \Rightarrow\ Y] \\
\Leftarrow & \quad \{\text{(c1) in the form } [S;\sim C \Rightarrow \sim X]\} \\
 & \quad \{\text{(a1)}\} \\
 & [\sim X \ \wedge\ Z \ \Rightarrow\ Y] \\
= & \quad \{\text{pred. calc.}\} \\
 & [Z \ \Rightarrow\ \neg\sim X \ \vee\ Y] \, ,
\end{array}
$$

and here we encounter our second constraint to be imposed on Z, viz.

(2) $\quad [Z \Rightarrow \neg\sim X \vee Y]$.

(<u>End</u> of Re $[A \Rightarrow B]$.)

<u>Re $[B \Rightarrow A]$</u>

$$
\begin{array}{ll}
 & [B \Rightarrow A] \\
\Leftarrow & \quad \{(a2) \text{ with } W := B \text{ , heuristics as before}\} \\
 & [B \Rightarrow S] \wedge [B; \sim S \Rightarrow Z] \text{ ,}
\end{array}
$$

and we tackle the conjuncts separately.

- $$
\begin{array}{ll}
 & [B \Rightarrow S] \\
\Leftarrow & \quad \{(c0)\} \\
 & [B \Rightarrow C] \\
\Leftarrow & \quad \{(b0)\} \\
 & \text{true} \text{ .}
\end{array}
$$

- $[B; \sim S \Rightarrow Z]$

The reader who has recorded which of the givens (a), (b), and (c) have been used so far, will observe that (b1) is the only one that has not yet been used. It is a fair guess that (b1) has to play a rôle. But it contains Y and our demonstrandum does not! The question is how to drag an occurrence of Y into the picture. Requirement (2) of Z is the only possibility! If we put

(3) $\quad [Z \equiv Z' \wedge (\neg\sim X \vee Y)]$

we obtain

$$
\begin{array}{ll}
 & [B; \sim S \Rightarrow Z] \\
= & \quad \{(3)\} \\
 & [B; \sim S \Rightarrow Z' \wedge (\neg \sim X \vee Y)] \\
= & \quad \{\text{pred. calc.}\} \\
 & [B; \sim S \Rightarrow Z'] \quad \wedge \quad [B; \sim S \wedge \sim X \Rightarrow Y] .
\end{array}
$$

Again we tackle these conjuncts separately.

$$
\begin{array}{ll}
\bullet\bullet & [B; \sim S \Rightarrow Z'] \\
\Leftarrow & \quad \{(b0)\} \\
 & [C; \sim S \Rightarrow Z'] \\
\Leftarrow & \quad \{(c1)\} \\
 & [X \Rightarrow Z'] \quad .
\end{array}
$$

which gives us our third contraint to be imposed on Z, viz.

(4) $\quad [X \Rightarrow Z']$.

$$
\begin{array}{ll}
\bullet\bullet & [B; \sim S \wedge \sim X \Rightarrow Y] \\
\Leftarrow & \quad \{(b1)\} \\
 & [B; \sim S \wedge \sim X \Rightarrow B; \sim C]
\end{array}
$$

<u>Reason</u> We *must* use (b1) and the above step is the *only* possibility.
(<u>End</u> of Reason.)

Now we are left with an expression that is entirely formulated in a "B & C - nomenclature". In particular, it no longer contains a reference to A or Z, and none of the rules (b) and (c) allow us to reimport these names into the expression. This means that in showing its validity we will not encounter new constraints on Z. For us this is the main reason isolate it as a separate

<u>Lemma</u> $[B: \sim S \ \wedge \ \sim X \ \Rightarrow \ B: \sim C]$,

to be shown later. (Another reason to isolate it that there are so many different proofs for it, all of them relatively long, and none of them really fascinating, i.e. coming close to being ugly.)

(<u>End</u> of Re $[B \Rightarrow A]$.)

x x x

This concludes our design of a proof for $[A \equiv B]$ and it also concludes the most fascinating and most critical part of our proof for Bird's theorem. Of course, it remains to be shown that our constraints on Z, viz.

(1) $[Z \Rightarrow X]$

(2) $[Z \Rightarrow \neg \sim X \vee Y]$

(3) $[Z \equiv Z' \wedge (\neg \sim X \vee Y)]$

(4) $[X \Rightarrow Z']$.

admit of a solution that is a preorder. (Constraint (2) is subsumed in (3), but we leave it as is.) It was Lex Bijlsma who at this point observed that the constraints admit precisely 1 solution, viz.

$$[Z \equiv X \wedge (\neg \sim X \vee Y)] \quad ,$$

- proof left to the reader - . Now we show that this solution is a preorder.

Z is reflexive , i.e. $[J \Rightarrow Z]$

$$\begin{array}{ll} & Z \\ = & \quad \{(3)\} \\ & Z' \wedge (\neg \sim X \vee Y) \\ \Leftarrow & \quad \{(4) \text{ and pred.calc.}\} \\ & X \wedge Y \\ \Leftarrow & \quad \{X \text{ and } Y \text{ are reflexive}\} \\ & J \quad . \end{array}$$

Z is transitive , i.e. $[Z;Z \Rightarrow Z]$

$$\begin{array}{ll} & [Z;Z \Rightarrow Z] \\ = & \quad \{(3)\} \\ & [Z;Z \Rightarrow Z' \wedge (\neg \sim X \vee Y)] \\ = & \quad \{\text{pred.calc.}\} \\ & [Z;Z \Rightarrow Z'] \wedge [Z;Z \Rightarrow \neg \sim X \vee Y] \quad . \end{array}$$

As for the first conjunct we observe,

$$\begin{array}{ll} & Z' \\ \Leftarrow & \quad \{(4)\} \\ & X \\ \Leftarrow & \quad \{X \text{ is transitive}\} \\ & X;X \\ \Leftarrow & \quad \{(1)\} \\ & Z;Z \quad . \end{array}$$

As for the second conjunct we observe.

$$\begin{array}{ll} & [Z;Z \Rightarrow \neg \sim X \vee Y] \\ = & \quad \{\text{pred. calc.}\} \\ & [Z;Z \wedge \sim X \Rightarrow Y] \end{array}$$

$\Leftarrow$ { Y is transitive }
$[Z;Z \wedge \sim X \Rightarrow Y;Y]$
$\Leftarrow$ { the Grand Dedekind, see Appendix for the rule and its "symbol dynamics" }
$[\sim X;\sim Z \wedge Z \Rightarrow Y]$
$\wedge$
$[\sim Z;\sim X \wedge Z \Rightarrow Y]$

For the first of these conjuncts — the second is left to the reader — we have

$[\sim X;\sim Z \wedge Z \Rightarrow Y]$
$\Leftarrow$ { (1) in the form $[\sim Z \Rightarrow \sim X]$ }
$[\sim X;\sim X \wedge Z \Rightarrow Y]$
$\Leftarrow$ { X's transitivity in the form $[\sim X;\sim X \Rightarrow \sim X]$ }
$[\sim X \wedge Z \Rightarrow Y]$
$=$ { pred. calc. }
$[Z \Rightarrow \neg\sim X \vee Y]$
$=$ { (2) }
true ,

which concludes our demonstration of Z being transitive.

* * *

Finally, we give a proof of the lemma. It contains — surprisingly and disappointingly — one more appeal to the transitivity of X.

<u>Lemma</u> $[B;\sim S \wedge \sim X \Rightarrow B;\sim C]$

<u>Proof</u>

$[B;\sim S \wedge \sim X \Rightarrow B;\sim C]$

$\Leftarrow$ { the Grand Dedekind-rule }

$[\sim X ; S \quad \wedge \quad B \quad \Rightarrow \quad B]$

$\wedge$

$[\sim B ; \sim X \quad \wedge \quad \sim S \quad \Rightarrow \quad \sim C]$

$=$ { the first conjunct "vanishes", the second is transposed }

$[X ; B \quad \wedge \quad S \quad \Rightarrow \quad C]$

$\Leftarrow$ { (c2) with $W := X ; B \quad \wedge \quad S$ }

$[X ; B \wedge S \quad \Rightarrow \quad S]$

$\wedge$

$[(X ; B \quad \wedge \quad S) ; \sim S \quad \Rightarrow \quad X]$

The first conjunct vanishes. For the second one we observe

$(X ; B \quad \wedge \quad S) ; \sim S$

$\Rightarrow$ { pred. calc. }

$X ; B ; \sim S$

$\Rightarrow$ { (b0) }

$X ; C ; \sim S$

$\Rightarrow$ { (c1) }

$X ; X$

$\Rightarrow$ { X is transitive }

X .

(<u>End</u> of Proof.)

<u>Remark</u> As mentioned before, the above lemma admits of many different proofs. All previous proofs we had lavishly used that S, B, and C are leftconditions, but the above proof - which was developed while being written down - does not use these facts at all. The net result is that our proof of Bird's theorem nowhere uses the givens that S, A, B, and C are leftconditions:

they only enter the picture for the benefit of translating the original problem statement into the relational notation. This surprise at the very end of this note will be food for further thought, and perhaps necessitate a rewrite. Peccavimus.
(End of Remark.)

This concludes our derivation of a proof for Bird's theorem.

* * *

Final Remarks

We wish to describe the flavour of our first effort to prove the theorem in order to understand better why that effort failed. Formulae (a) define A as the weakest relation satisfying

(a0) $\quad [A \Rightarrow S]$ and

(a1) $\quad [A ; \sim S \Rightarrow Z]$.

We can give a closed expression for that relation by rewriting (a1) as follows

$$
\begin{array}{ll}
 & [A ; \sim S \Rightarrow Z] \\
= & \quad \{\text{left-exchange}\} \\
 & [\neg Z ; S \Rightarrow \neg A] \\
= & \quad \{\text{contrapositive}\} \\
 & [A \Rightarrow \neg(\neg Z ; S)] \; .
\end{array}
$$

The closed expression for A now is

$$[A \equiv S \wedge \neg(\neg Z ; S)] \quad .$$

Similarly, we have closed expressions for B and C, viz.

$$[B \equiv C \wedge \neg(\neg Y ; C)]$$

$$[C \equiv S \wedge \neg(\neg X ; S)] \quad .$$

Theorem $[A \equiv B]$ can now be formulated without any reference to the auxiliary names A, B, or C, namely as: for all S

(x) $$[S \wedge \neg(\neg Z ; S) \equiv S \wedge \neg(\neg X ; S) \wedge \neg(\neg Y ; (S \wedge \neg(\neg X ; S)))] \quad ,$$

and our task is to solve it for Z, using the relational calculus as our exclusive tool. As mentioned before, we did not succeed. Presumably, the prime reason for our failure is that expression (x) and its manipulative descendants are so long and so fine-grained that they offer far too many manipulative possibilities and far too little guidance for how to continue calculation. Especially in an exercise like the current one, where —in retrospect— the solution space is so tight, an abundance of manipulative freedom at the same time implies an abundance of dead alleys. (In our first effort, the situation was even worse because we had coupled sets to monotypes, rendering (x) more complicated by almost one order of magnitude.)

The method that we then applied by attaching auxiliary names to the subexpressions of (x) and by completely spelling out their properties, did the job. It was only then that it became apparent how little freedom we had in proving Bird's theorem. (At a later stage Perry Moerland – student member of the ETAC – successfully repeated the experiment for the monotype representation of sets.)

A viable, quite different way to prove Bird's theorem was exemplified by Henk Doornbos – graduate student to R.C. Backhouse – He introduces a binary operator "/", defined by

$$[\ P/Q \ \equiv\ \neg(\neg P ; Q)\ ]\ ,$$

in terms of which (x) can be rewritten as, for all S

$$[\ S \wedge Z/S \ \equiv\ S \wedge X/S \ \wedge\ Y/(S \wedge X/S)\ ]\ .$$

With the purpose of solving this for Z, he then develops – in a rather goal directed fashion – a theory of "/". This method is viable especially when – as in Doornbos's case – the interest in operator "/" goes beyond the current exercise.

* * *

A final Final Remark concerns an investigation of Lex Bijlsma's. He has shown, using a pointwise argument, that $X \wedge (\neg{\sim}X \vee Y)$ is the only candidate relation that can solve Bird's problem. It is nice for us that, with our pointless reasoning, we have encountered this solution, but at the same time it is a pity that we have not received any signal of its uniqueness.

(End of Final Remarks.)

Acknowledgements

In the first place we would like to acknowledge Richard S. Bird for communicating such a fascinating, easily stated little problem. This text is written for and dedicated to him, mainly.

Secondly, we would like to thank Jaap van der Woude who - after our initial failure - showed us that the theorem can be proven within the relational calculus as we know it.

Finally, we wish to express our gratitude for the many critical remarks of Lex Bijlsma, Ronald Bulterman, Netty van Gasteren, Rob Hoogerwoord, Perry Moerland, John Segers, and Carel Scholten, all members of the ETAC.

(End of Acknowledgements.)

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7 November 1991,
Eindhoven.

Appendix

All rules of the relational calculus that we used in this note can be derived from the following set of postulates

- $[\sim x \Rightarrow y] \equiv [x \Rightarrow \sim y]$

- $;$ is associative

- $[x ; y \Rightarrow z] \equiv [\neg z ; \sim y \Rightarrow \neg x]$
  $[x ; y \Rightarrow z] \equiv [\sim x ; \neg z \Rightarrow \neg y]$,
  the left- and right-exchange rules

- $[J ; x \equiv x]$.

The most important consequences are

- $\sim$ distributes over all logical expressions,
  $[x] \equiv [\sim x]$, $[\sim\sim x \equiv x]$
  $[\sim true \equiv true]$, $[\sim false \equiv false]$,
  $[\sim (x ; y) \equiv \sim y ; \sim x]$

- $;$ is universally disjunctive in both arguments, and hence monotonic

- the Grand Dedekind rule :

$$[x;y \wedge z \Rightarrow r;s]$$
$$\Leftarrow$$
$$[z;\sim y \wedge x \Rightarrow r]$$
$$\wedge$$
$$[\sim x;z \wedge y \Rightarrow s] \quad .$$

Its "symbol dynamics" – an aide-memoire towards remembering the rule – is as follows. It is a rule for separating the two operands in $r;s$. To separate the *left* operand, the *left* operand in $x;y$ is swapped with conjunct z and operand in $x;y$ that remains is transposed. The separate the *right* operand of $r;s$ the *right* operand of $x;y$ is swapped and the remaining one transposed.

This rule, which is very useful for the practice of relational calculation, has been designed by Henk Doornbos and appears in "A relational theory of data types", a technical report by R. Backhouse, E. Voermans, and J. van der Woude.

- There are two definitions of x being a leftcondition, a weak and a strong one:

  $[x;true \Rightarrow x]$ weak

  $[x;true \equiv x]$ strong

- x is leftcondition $\equiv$ $\sim x$ is rightcondition

(*End* of Appendix.)

(*End* of WF147.)