A beautiful proof of an ugly theorem
(from the relational calculus)

The ugly theorem from the relational calculus is that for any relations x, y, and z we have

(0) $[\ x;y \ \wedge\ z \ \Rightarrow\ x;(y \wedge\ \sim x;z)\ ]$.

The theorem was communicated to us by Netty van Gasteren near the end of last Tuesday's session of the ETAC. But besides that, she also proposed to tackle (0) by investigating equation "E: (1)", with (1) given by

(1) $[\ x;y \ \wedge\ z \ \Rightarrow\ x;E\ ]$.

The proposal to explore (1) rather than (0) offers at least two advantages. One of them is that the transition from (0) to (1) does away with most of the unwieldiness of (0) by concentrating on a more macroscopic structure of the formula. The benefits of such shifts of attention are well-known in the meantime. A second advantage of the transition is that it gives an answer to the question of how — for Heaven's Sake — someone can invent a theorem like (0). The answer is: by exploring equations like (1). (The reasons for being interested in (1) are not of our concern, right now.)

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So now let's investigate (1) and try to solve it for E. First we use the rule

$$[a \Rightarrow b] \equiv (\underline{A}h:: [b \Rightarrow h] \Rightarrow [a \Rightarrow h]),$$

to rewrite (1) into the form:

for any h,

(2) $\quad [x;E \Rightarrow h] \quad \Rightarrow \quad [x;y \wedge z \Rightarrow h]$.

The reason for doing this is that the relational calculus offers nice rewrite rules for compositions occurring in an antecedent, and hardly any rules for compositions in a consequent.

Next we tackle the two sides of (2) separately:

antecedent of (2)
$=\quad$ {see (2)}
$[x;E \Rightarrow h]$
$=\quad$ {rule of rotation}
$[\sim x;\neg h \Rightarrow \neg E]$, $\quad$ (*)

and

consequent of (2)
$=\quad$ {see (2)}
$[x;y \wedge z \Rightarrow h]$
$=\quad$ {pred. calc.}

$[x;y \Rightarrow h \vee \neg z]$

$=$ {rule of rotation ande de Morgan}

$[\sim x ; (\neg h \wedge z) \Rightarrow \neg y]$.

$\Leftarrow$ { monotonicity of ; . The antecedent of (*) pops up! }

$[(\sim x ; \neg h) \wedge (\sim x ; z) \Rightarrow \neg y]$

$=$ {pred. calc.}

$[\sim x ; \neg h \Rightarrow \neg y \vee \neg(\sim x ; z)]$ (**)

Now (**) $\Leftarrow$ (*) , provided we choose E such that

$[\neg y \vee \neg(\sim x ; z) \Leftarrow \neg E]$,

or, equivalently,

$[y \wedge (\sim x ; z) \Rightarrow E]$.

The strongest E satisfying this, renders (1) into theorem (0).

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With compliments to Netty,

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