A homework assignment by Jaap van der Woude

Jaap van der Woude told me the following theorem from the relational calculus:

(0) for precondition p,

$$[(x \wedge p) ; y \;\equiv\; x ; (\sim p \wedge y)] \;.$$

Without him saying so explicitly, I understood that now he wanted me to prove the theorem for myself. And rightly so.

We will not just give the proof, but we also describe which considerations led to it. First we see an equivalence with two equally complicated sides. Now we have two options, to wit embark on an equivalence preserving transformation starting at either side of the equivalence, or give a proof by mutual implication. The latter creates more manipulative possibilities, in particular applications of the Fancy Rule of Rotation, but that does not look very appealing in view the compositions at both sides of (0). (This may be a judgement of an amateur.) So, we try to get away with the equivalence preserving transformation, which is to be preferred anyway.

The first steps of that calculation are

$$(x \wedge p) ; y$$
$$= \quad \{\text{pred. calc.}\}$$

$$
\begin{array}{ll}
 & (x \wedge p) \,;\, (true \wedge y) \\
= & \quad \{ \text{ pred. calc. } \} \\
 & (x \wedge p) \,;\, (\cdots \vee \cdots) \wedge y
\end{array}
$$

Why? The central symbol in $(x \wedge p) ; y$ is the semicolon, and one of the equivalence preserving transformations involving the semicolon is that it distributes over disjunction. Hence.

Next we have to decide on the disjuncts. It is very tempting to choose $\sim p$ for one of them, because that will create subexpression $\sim p \wedge y$, which occurs in the right-hand side of (0). For the other disjunct we have many possibilities left, but we choose $\neg \sim p$, because a simple way to take into account that p is a precondition is by the rules (1) or (2)

(1) $\quad [\, p ; \neg \sim p \;\equiv\; false \,]$

(2) $\quad [\, \neg p ; \sim p \;\equiv\; false \,]$.

Thus we obtain

$$
\begin{array}{ll}
 & (x \wedge p) \,;\, ((\neg \sim p \vee \sim p) \wedge y) \\
= & \quad \{ \wedge \text{ over } \vee \text{ , and } ; \text{ over } \vee \} \\
 & (x \wedge p) \,;\, (\neg \sim p \wedge y) \\
 & \quad \vee \;\; (x \wedge p) \,;\, (\sim p \wedge y) \\
= & \quad \{ \text{ the first disjunct } \equiv \text{ false,} \\
 & \qquad \text{by (1) and monotonicity of } ; \} \\
(*) & (x \wedge p) \,;\, (\sim p \wedge y) \;.
\end{array}
$$

By the same token the right-hand side of (0) is massaged into expression (x). This settles the theorem.

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So the standard heuristics worked out well, again. Meanwhile, I wonder whether Van der Woude has a shorter proof. In any case, he is welcome to grade me, be it that he must choose between an A or a B.

W.H.J. Feijen,
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