PREDICATE CALCULUS – PART 0

- Operators in the order of decreasing binding power:

$.$	functional application
$\neg$	negation
$\wedge$, $\vee$	conjunction, disjunction
$\Rightarrow$, $\Leftarrow$	implication, consequence
$\equiv$, $\not\equiv$	equivalence, difference

- Leibniz's Rule

$$[x = y] \;\Rightarrow\; [f.x = f.y]$$

- Associativity

Equivalence, difference, conjunction, and disjunction are associative. Equivalence and difference are mutually associative. Functional application is left-associative, i.e.

$$[\, f.x.y = (f.x).y \,] \quad .$$

- Symmetry and Idempotence

Equivalence, difference, conjunction, and disjunction are symmetric. Conjunction and disjunction are idempotent.

- Unit elements and zero elements

Units:

$$[\, X \equiv true \equiv X \,]$$
$$[\, X \not\equiv false \equiv X \,]$$
$$[\, X \wedge true \equiv X \,]$$
$$[\, X \vee false \equiv X \,]$$

Zeroes:

$$[\, X \wedge false \equiv false \,]$$
$$[\, X \vee true \equiv true \,]$$

- <u>Elementary Rules</u>

Distribution

$$[X \vee (Y \equiv Z) \equiv X \vee Y \equiv X \vee Z]$$
$$[X \wedge (Y \equiv Z) \equiv X \wedge Y \equiv X \wedge Z \equiv X]$$
$$[X \vee (Y \wedge Z) \equiv (X \vee Y) \wedge (X \vee Z)]$$
$$[X \wedge (Y \vee Z) \equiv (X \wedge Y) \vee (X \wedge Z)]$$

Absorption

$$[X \vee (X \wedge Y) \equiv X]$$
$$[X \wedge (X \vee Y) \equiv X]$$
$$[X \vee Y \Leftarrow X]$$
$$[X \wedge Y \Rightarrow X]$$

Complement

$$[X \vee (\neg X \wedge Y) \equiv X \vee Y]$$
$$[X \wedge (\neg X \vee Y) \equiv X \vee Y]$$

Golden Rule

$$[X \wedge Y \equiv X \equiv Y \equiv X \vee Y]$$

Implication

$$[X \Rightarrow Y \equiv X \vee Y \equiv Y]$$
$$[X \Rightarrow Y \equiv X \wedge Y \equiv X]$$
$$[X \Rightarrow Y \equiv \neg X \vee Y]$$
$$[X \Leftarrow Y \equiv Y \Rightarrow X]$$

Negation

$$[\neg (X \equiv Y) \equiv \neg X \equiv Y]$$
$$[\neg X \vee X \equiv \mathit{true}]$$
$$[\neg\neg X \equiv X]$$
$$[\neg (X \vee Y) \equiv \neg X \wedge \neg Y]$$
$$[\neg (X \wedge Y) \equiv \neg X \vee \neg Y]$$

- Exercises

0. $[P \wedge (X \equiv Y \equiv Z) \equiv P \wedge X \equiv P \wedge Y \equiv P \wedge Z]$
1. $[X \wedge (X \equiv Y) \equiv X \wedge Y]$
2. $[(X \equiv X \wedge Y) \vee (Y \equiv X \wedge Y)]$
3. $[X \wedge (X \Rightarrow Y) \equiv X \wedge Y]$
4. $[X \wedge Y \Rightarrow Z \equiv X \Rightarrow (Y \Rightarrow Z)]$
5. $[(X \Rightarrow Y) \wedge (Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z)]$
6. $[(X \Rightarrow Y) \vee (Y \Rightarrow Z)]$
7. $[(X \Rightarrow Y) \wedge (Y \Rightarrow X) \equiv X \equiv Y]$
8. $[X \Rightarrow true]$
9. $[true \Rightarrow X \equiv X]$
10. $[(X \equiv Y) \Rightarrow (X \Rightarrow Y)]$
11. $[X \vee Y \Rightarrow Z \equiv (X \Rightarrow Z) \wedge (Y \Rightarrow Z)]$
12. $[X \Rightarrow Y \wedge Z \equiv (X \Rightarrow Y) \wedge (X \Rightarrow Z)]$
13. $[X \wedge Y \Rightarrow Z \equiv (X \Rightarrow Z) \vee (Y \Rightarrow Z)]$
14. $[X \Rightarrow Y \vee Z \equiv (X \Rightarrow Y) \vee (X \Rightarrow Z)]$
15. $[X \Rightarrow (Y \equiv Z) \equiv X \Rightarrow Y \equiv X \Rightarrow Z]$
16. $[(X \equiv Y) \Rightarrow Z \equiv X \Rightarrow Z \equiv Y \Rightarrow Z \equiv Z]$
17. $[(X \equiv Y \equiv Z) \Rightarrow P \equiv X \Rightarrow P \equiv Y \Rightarrow P \equiv Z \Rightarrow P]$
18. $[(X \Rightarrow Y) \Rightarrow ((Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z))]$
19. $[(Y \Rightarrow Z) \Rightarrow ((X \Rightarrow Y) \Rightarrow (X \Rightarrow Z))]$
20. $[X \Rightarrow (Y \Rightarrow Z) \equiv Y \Rightarrow (X \Rightarrow Z)]$
21. $[X \Rightarrow (Y \Rightarrow Z) \equiv (X \Rightarrow Y) \Rightarrow (X \Rightarrow Z)]$
22. $[X \Rightarrow (Y \Rightarrow Z) \equiv X \wedge Y \Rightarrow X \wedge Z]$
23. $[X \vee (Y \Rightarrow Z) \equiv X \vee Y \Rightarrow X \vee Z]$
24. $[(X \Rightarrow Y) \Rightarrow (X \vee Z \Rightarrow Y \vee Z)]$

25. $[(X \Rightarrow Y) \Rightarrow (X \wedge Z \Rightarrow Y \wedge Z)]$

26. $[((X \Rightarrow Y) \Rightarrow Z) \Rightarrow (X \Rightarrow (Y \Rightarrow Z))]$

27. $[\text{false} \Rightarrow X]$

28. $[X \Rightarrow \text{false} \equiv \neg X]$

29. $[\neg X \equiv X \equiv \text{false}]$

30. $[\neg X \Rightarrow X \equiv X]$

31. $[X \Rightarrow Y \equiv \neg X \Leftarrow \neg Y]$

32. $[X \wedge Y \equiv X \wedge Z \equiv \neg X \vee (Y \equiv Z)]$

33. $[X \wedge \neg Y \equiv Y \wedge \neg X \equiv X \equiv Y]$

34. $[X \wedge (Y \vee Z) \equiv X \wedge Y \equiv (X \wedge Z \Rightarrow Y)]$

35. $[X \vee (Y \wedge Z) \equiv X \vee Y \equiv (X \vee Z \Leftarrow Y)]$

36. $[(X \wedge Y) \vee (\neg X \wedge Z) \equiv (\neg X \vee Y) \wedge (X \vee Z)]$

37. $[[X]] \equiv [X]$

38. $[X \Rightarrow Y] \Rightarrow ([X] \Rightarrow [Y])$

39. $[X] \vee [Y] \Rightarrow [X \vee Y]$

40. $[X] \wedge [Y] \equiv [X \wedge Y]$

41. $[X \equiv Y] \Rightarrow ([X] \equiv [Y])$

PREDICATE CALCULUS – PART 1

- <u>Elementary Rules</u>

Trading

$$[\, (\underline{A}x : r.x \wedge s.x : p.x) \equiv (\underline{A}x : r.x : \neg s.x \vee p.x) \,]$$
$$[\, (\underline{E}x : r.x \wedge s.x : p.x) \equiv (\underline{E}x : r.x : s.x \wedge p.x) \,]$$

Splitting the range

$$[\, (\underline{A}x : r.x \vee s.x : p.x) \equiv (\underline{A}x : r.x : p.x) \wedge (\underline{A}x : s.x : p.x) \,]$$
$$[\, (\underline{E}x : r.x \vee s.x : p.x) \equiv (\underline{E}x : r.x : p.x) \vee (\underline{E}x : s.x : p.x) \,]$$

Distribution

$$[\, Q \vee (\underline{A}x :: p.x) \equiv (\underline{A}x :: Q \vee p.x) \,]$$
$$[\, Q \wedge (\underline{E}x :: p.x) \equiv (\underline{E}x :: Q \wedge p.x) \,] ,$$

and for non-empty ranges :

$$[\, Q \wedge (\underline{A}x :: p.x) \equiv (\underline{A}x :: Q \wedge p.x) \,]$$
$$[\, Q \vee (\underline{E}x :: p.x) \equiv (\underline{E}x :: Q \vee p.x) \,] ,$$

and for boolean range :

$$[\, (\underline{A}x :: p.x) \,] \equiv (\underline{A}x :: [p.x])$$

One - point - rules

$$[\, (\underline{A}x : [x = y] : p.x) \equiv p.y \,]$$
$$[\, (\underline{E}x : [x = y] : p.x) \equiv p.y \,]$$

Dummy renaming

$$[\, (\underline{Q}x : r.x : p.x) \equiv (\underline{Q}y : r.y : p.y) \,]$$

- <u>Exercises</u>

0. $[\, (\underline{A}x :: p.x \wedge q.x) \equiv (\underline{A}x :: p.x) \wedge (\underline{A}x :: q.x) \,]$

1. $[\, (\underline{E}x :: p.x \vee q.x) \equiv (\underline{E}x :: p.x) \vee (\underline{E}x :: q.x) \,]$

2. $[(\underline{A}x :: p.x \equiv q.x)$
$\quad \Rightarrow \quad (\underline{A}x :: p.x) \equiv (\underline{A}x :: q.x)\ ]$

3. $[(\underline{A}x :: p.x \equiv q.x)$
$\quad \Rightarrow \quad (\underline{E}x :: p.x) \equiv (\underline{E}x :: q.x)\ ]$

4. $[(\underline{A}x :: p.x \Rightarrow q.x)$
$\quad \Rightarrow \quad (\underline{A}x :: p.x) \Rightarrow (\underline{A}x :: q.x)\ ]$

5. $[(\underline{A}x :: p.x \Rightarrow q.x)$
$\quad \Rightarrow \quad (\underline{E}x :: p.x) \Rightarrow (\underline{E}x :: q.x)\ ]$

6. $[(\underline{A}x :: p.x) \Rightarrow (\underline{A}x :: p.x \vee q.x)\ ]$

7. $[(\underline{E}x :: p.x) \Rightarrow (\underline{E}x :: p.x \vee q.x)\ ]$

8. $[(\underline{A}x : r.x \vee s.x : p.x) \Rightarrow (\underline{A}x : r.x : p.x)\ ]$

9. $[(\underline{E}x : r.x \vee s.x : p.x) \Leftarrow (\underline{E}x : r.x : p.x)\ ]$

10. $[(\underline{A}x :: p.x) \vee (\underline{A}x :: q.x) \Rightarrow (\underline{A}x :: p.x \vee q.x)\ ]$

11. $[(\underline{E}x :: p.x) \wedge (\underline{E}x :: q.x) \Leftarrow (\underline{E}x :: p.x \wedge q.x)\ ]$

12. $[(\underline{A}x : r.x : p.x) \equiv (\underline{A}x : r.x : r.x \wedge p.x)\ ]$

13. $[(\underline{E}x : r.x : p.x) \equiv (\underline{E}x : r.x : r.x \Rightarrow p.x)\ ]$

14. $[\ Q \Rightarrow (\underline{A}x :: p.x) \equiv (\underline{A}x :: Q \Rightarrow p.x)\ ]$

15. $[\ (\underline{E}x :: p.x) \Rightarrow Q \equiv (\underline{A}x :: p.x \Rightarrow Q)\ ]$

16. $[(\underline{A}x : r.x : r.x \vee p.x) \equiv true\ ]$

17. $[(\underline{E}x : r.x : \neg r.x \wedge p.x) \equiv false\ ]$

18. $[(\underline{A}y : r.y : (\underline{A}x : r.x : p.x) \Rightarrow p.y)\ ]$

19. $[(\underline{A}y : r.y : p.y \Rightarrow (\underline{E}x : r.x : p.x))]$

20. $[(\underline{A}x :: Q \wedge p.x) \equiv (Q \wedge (\underline{A}x :: p.x)) \vee (\underline{A}x :: false)\ ]$

21. $[(\underline{E}x :: Q \vee p.x) \equiv (Q \vee (\underline{E}x :: p.x)) \wedge (\underline{E}x :: true)\ ]$

22. For non-empty range,
$[(\underline{A}x :: p.x) \Rightarrow Q \equiv (\underline{E}x :: p.x \Rightarrow Q)\ ]$

23. For non-empty range,
$[\ Q \Rightarrow (\underline{E}x :: p.x) \ \equiv \ (\underline{E}x :: Q \Rightarrow p.x)\ ]$

24. $[\ (\underline{A}y : r.y : (\underline{E}x :: r.x))\ ]$

25. $[\ (\underline{A}x :: p.x \vee q.x) \Rightarrow (\underline{E}x :: p.x) \vee (\underline{A}x :: q.x)\ ]$

26. $[\ (\underline{A}x :: p.x) \Rightarrow (\underline{E}x :: p.x) \ \equiv \ (\underline{E}x :: true)\ ]$

27. $[\ (\underline{E}x :: p.x) \Rightarrow (\underline{A}x :: p.x) \ \equiv \ (\underline{A}x,y :: p.x \equiv q.x)\ ]$

28. $[\ (\underline{A}x : r.x \wedge [x=y] : p.x) \ \equiv \ \neg r.y \vee p.y\ ]$

29. $[\ (\underline{E}x : r.x \wedge [x=y] : p.x) \ \equiv \ r.y \wedge p.y\ ]$

30. $[\ (\underline{A}x :: (\underline{E}y :: p.x.y)) \ \vee \ (\underline{A}y :: (\underline{E}x :: \neg p.x.y))\ ]$

31. $[\ (\underline{A}x,y : r.y \wedge [x=y] : p.x.y) \ \equiv \ (\underline{A}y : r.y : p.y.y)\ ]$

32. $[\ (\underline{A}x : 0 \le x \wedge x < y+1 : p.x)$
$\equiv (\underline{A}x : 0 \le x \wedge x < y : p.x) \wedge (y < 0 \vee p.y)\ ]$

33. $[\ (\underline{E}x : 0 \le x \wedge x < y+1 : p.x)$
$\equiv (\underline{E}x : 0 \le x \wedge x < y : p.x) \vee (0 \le y \wedge p.y)\ ]$

34. $[\ (\underline{A}x,y : r.x.y \wedge x < y+1 : p.x.y)$
$\equiv (\underline{A}x,y : r.x.y \wedge x < y : p.x.y) \wedge (\underline{A}x : r.x.x : p.x.x)\ ]$

35. $[\ (\underline{A}x : 0 \le x : 1 \le x) \ \equiv \ false\ ]$

36. $[\ (\underline{A}y :: y \le 0 \Rightarrow (\underline{A}x : 0 \le x \wedge x < y : p.x))\ ]$

37. Rewrite $(\underline{A}x : 0 \le x \wedge x \le y : x < |y|)$ into an equivalent expression that no longer contains a quantifier

38. $[\ (\underline{A}x,y : x < y : (\underline{A}z :: x < z \vee z < y))\ ]$

39. Find all left-zero, right-zero, left-unit, and right-unit elements of operator $\Rightarrow$.

40. Prove that a conjunctive (disjunctive) predicate transformer is monotonic.

41. Prove that, for monotonic f,
$[\ f.(\underline{A}X :: X) \Rightarrow (\underline{A}X :: f.X)\ ]$, and
$[\ (\underline{E}X :: f.X) \Rightarrow f.(\underline{E}X :: X)\ ]$.

42. $[X \Rightarrow Y] \equiv (\underline{A} H :: [H \Rightarrow X] \Rightarrow [H \Rightarrow Y])$

43. $[X \Rightarrow Y] \equiv (\underline{A} H :: [Y \Rightarrow H] \Rightarrow [X \Rightarrow H])$

44. $[X \equiv Y] \equiv (\underline{A} H :: [H \vee X] \equiv [H \vee Y])$

45. Prove that predicate transformers f and g satisfying

$[f.X \vee Y] \equiv [X \vee g.Y]$,

for all X and Y, are universally conjunctive.

MISCELLANEOUS

- A little calculus of Δ and ∇

For any two numbers x and y we deem to be defined the expressions $x \Delta y$ and $x \nabla y$. They satisfy — by postulate —

$$[\ z \geq x \Delta y \ \equiv\ z \geq x \ \wedge\ z \geq y\ ]$$

$$[\ z \leq x \nabla y \ \equiv\ z \leq x \ \wedge\ z \leq y\ ] \quad ;$$

here, the square brackets denote universal quantification over x, y, and z. We now can prove a lot of properties, relating Δ, ∇, and other arithmetic operators.

0. Addition distributes over both Δ and ∇.

1. Δ and ∇ are associative, symmetric, and idempotent

2. $[\ x \Delta y \geq x \ \wedge\ x \Delta y \geq y\ ]$
   $[\ x \nabla y \leq x \ \wedge\ x \nabla y \leq y\ ]$

3. $[\ x \Delta y = x \ \vee\ x \Delta y = y\ ]$
   $[\ x \nabla y = x \ \vee\ x \nabla y = y\ ]$

4. $[\ z \leq x \Delta y \ \equiv\ z \leq x \ \vee\ z \leq y\ ]$
   $[\ z \geq x \nabla y \ \equiv\ z \geq x \ \vee\ z \geq y\ ]$

5. Δ and ∇ distribute over each other.

6. $[\ -(x \Delta y) \ =\ (-x) \nabla (-y)\ ]$
   $[\ -(x \nabla y) \ =\ (-x) \Delta (-y)\ ]$

7. $[\ x \Delta y = x \nabla y \ \equiv\ x = y\ ]$

8. $[\ x \Delta y = x \ \equiv\ y = x \nabla y\ ]$

9. $[\ x \Delta y = x \ \equiv\ x \geq y\ ]$

10. $[\ x \nabla y = x \ \equiv\ x \leq y\ ]$

11. $[\ z \geq 0 \ \Rightarrow\ z \times (x \Delta y) = (z \times x) \Delta (z \times y)\ ]$

And so on.

Sometimes it is useful to have on hand the two outlandish, non-numeric values "pinf" and "minf", which satisfy

$[x \vartriangle \text{pinf} = \text{pinf}]$ $\quad$ $[x \vartriangle \text{minf} = x]$

$[x \triangledown \text{pinf} = x]$ $\quad$ $[x \triangledown \text{minf} = \text{minf}]$.

A highly common interpretation is <u>max</u> for $\vartriangle$ and <u>min</u> for $\triangledown$. We shall stick to this interpretation and notation in what follows.

- <u>On MAX and MIN</u>

For finite and (mostly) nonempty ranges $r.x$ and for integer expressions $f.x$, we will consider expressions

$(\underline{MAX}\, x : r.x : f.x)$ and $(\underline{MIN}\, x : r.x : f.x)$.

They can be defined by (, omitting the range)

$[\ z \geq (\underline{MAX}\, x :: f.x) \ \equiv\ (\underline{A}\, x :: z \geq f.x)\ ]$

$[\ z \leq (\underline{MIN}\, x :: f.x) \ \equiv\ (\underline{A}\, x :: z \leq f.x)\ ]$.

They inherit many properties from universal quantification and from <u>max</u> and <u>min</u>. We will list some, for <u>MAX</u> mainly.

12. Splitting the range

$[(\underline{MAX}\, x : r.x \vee s.x : f.x)$
$= (\underline{MAX}\, x : r.x : f.x)\ \underline{max}\ (\underline{MAX}\, x : s.x : f.x)\ ]$

13. Splitting the term

$[\ (\underline{MAX}\, x :: f.x\ \underline{max}\ g.x)$
$= (\underline{MAX}\, x :: f.x)\ \underline{max}\ (\underline{MAX}\, x :: g.x)\ ]$

14. One-point-rule

$[\ (\underline{MAX}\, x : [x = y] : f.x) = f.y\ ]$

15. Distribution

$[\ (\underline{MAX}\, x :: f.x\ \underline{min}\ g.y) = (\underline{MAX}\, x :: f.x)\ \underline{min}\ g.y\ ]$

$[\ (\underline{MAX}\, x :: f.x\ \underline{max}\ g.y) = (\underline{MAX}\, x :: f.x)\ \underline{max}\ g.y\ ]$

$[\ (\underline{MAX}\, x :: f.x + g.y) = (\underline{MAX}\, x :: f.x) + g.y\ ]$

It should be noted that latter two rules hold for non-empty ranges only.

16. $[\ z = (\underline{MAX}\, x :: f.x)$
$\equiv\ (\underline{E}\, x ::\ z = f.x)\ \wedge\ (\underline{A}\, x ::\ f.x \leq z)\ ]$

17. $[\ f.z = (\underline{MAX}\, x : r.x : f.x)$
$\equiv\ r.z\ \wedge\ (\underline{A}\, x : r.x : f.x \leq f.z)\ ]$

18. $[\ -(\underline{MAX}\, x :: f.x) = (\underline{MIN}\, x :: -f.x)\ ]$

Etcetera.

• <u>On S and N</u>

For finite ranges r.x and integer expression f.x, we will consider expressions

$(\underline{S}\, x : r.x : f.x)$ and $(\underline{N}\, x : r.x)$

For the former we have — by definition —

$[\ (\underline{S}\, x : false : f.x) = 0\ ]$
$[\ (\underline{S}\, x : [x = y] : f.x) = f.y\ ]$
$[\ (\underline{S}\, x : r.x \vee s.x : f.x)$
$= (\underline{S}\, x : r.x : f.x) + (\underline{S}\, x : s.x : f.x)$
$- (\underline{S}\, x : r.x \wedge s.x : f.x)\ ]$

Furthermore, all familiar rules about finite summation apply.

The latter is defined by

$[\ (\underline{N}\, x : r.x) = (\underline{S}\, x : r.x : 1)\ ]$,

so that it is natural-valued. For the corresponding one-point-rule we will often use the format

$[\ (\underline{N}\, x : [x = y] \wedge r.x)$
$= \ \underline{if}\ r.y \rightarrow 1$
$[\!]\ \neg r.y \rightarrow 0$
$\underline{fi}$
$]$

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