A different derivation of the Schoenmakers - Kaldewaij Linear Search (for the record)

Quite recently B. Schoenmakers and A. Kaldewaij – two colleagues at Eindhoven University – revealed a new and very nice Linear Search algorithm. Their algorithm is so nice because it has an extremely simple formal specification and because its application to a class of well-known little programming problems turns out to be quite effective and – surprisingly so – to yield new and elegant little programs. The purpose of this note is to derive the algorithm of Schoenmakers and Kaldewaij by not resorting to set notation, as they did.

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We are given a boolean function S, which is defined on the nonempty natural segment $0 \leq i \wedge i \leq N$. The algorithm constructs a solution of equation $i : 0 \leq i \wedge i \leq N \wedge S.i$, known to exist. More formally, program SK-search has to meet specification

$$\{0 \leq N \wedge (\underline{E}i : 0 \leq i \wedge i \leq N : S.i)\}$$

SK-search

$$\{0 \leq x \wedge x \leq N \wedge S.x\}$$

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Our first step is to massage the post-condition so as to make it resemble the precondition. By applying the one-point-rule we obtain for the postcondition

$$0 \leq x \;\wedge\; x \leq N \;\wedge\; (\mathbf{E}\, i : x \leq i \wedge i \leq x : S.i) \;.$$

Our second step consists in the introduction of a shorthand for the quantified expressions so as to achieve brevity. With H defined by

$$H.x.y \;\equiv\; (\mathbf{E}\, i : x \leq i \wedge i \leq y : S.i) \;,$$

the specification can be rewritten as

$\{0 \leq N \;\wedge\; H.0.N\}$
  SK-search
$\{0 \leq x \;\wedge\; x \leq N \;\wedge\; H.x.x\}$.

The third step is the decision to implement SK-search by a repetition with invariant

P: $\quad 0 \leq x \wedge x \leq y \wedge y \leq N \wedge \; H.x.y$

and variant function $y - x$. Thus we can obtain a program of the form

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  x, y := 0, N  {P}
; do x ≠ y
  →  {x < y  ∧  P}
     if  B  →  x := x + 1
     ▯   C  →  y := y - 1
     fi
     {P}
  od
  {x = y  ∧  P}  .
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The fourth and last step is to find boolean expressions B and C such that abortion of the alternative construct is excluded and the invariance of P is guaranteed indeed. For the invariance of P's last conjunct – taking the remaining ones for granted – we require B to satisfy

$$B \wedge H.x.y \Rightarrow H.(x+1).y \qquad \text{for } x<y ,$$

or equivalently, factoring out B,

$$B \Rightarrow (H.x.y \Rightarrow H.(x+1).y) \qquad \text{for } x<y .$$

In order to find a suitable B, we start manipulating the consequent

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\begin{array}{lll}
 & & H.x.y \Rightarrow H.(x+1).y \\
 & = & \{\text{definition of } H, \text{ using } x<y\} \\
 & & S.x \vee H.(x+1).y \Rightarrow H.(x+1).y \\
 & = & \{\text{predicate calculus}\} \\
 & & S.x \Rightarrow H.(x+1).y \\
 & = & \{\text{predicate calculus}\} \\
 & & \neg S.x \vee H.(x+1).y \\
(*) & \Leftarrow & \{\text{definition of } H, \text{ using } x<y, \text{ i.e } x+1 \leq y\} \\
 & & \neg S.x \vee S.y \quad .
\end{array}
$$

As a result $\neg S.x \vee S.y$ is a candidate for B. By symmetry, $\neg S.y \vee S.x$ is a candidate for C. With these choices abortion of the alternative construct is excluded as well. It was with this latter requirement in mind that we have chosen the strengthening as recorded in step (*).

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