A calculation hinted at by dr. Lincoln A. Wallen

For predicates p and q (on the natural numbers) satisfying

(0) $[(\underline{A}x :: p.x \vee q.x)]$,

we shall show

(1) $[(\underline{A}x :: p.x \vee (\underline{E}y : y < x : \neg p.y \vee \neg q.y))$
$\vee$
$(\underline{A}x :: q.x \vee (\underline{E}y : y < x : \neg p.y \vee \neg q.y))$
$]$.

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Formula (1) is a quite complicated expression, the interior structure of which does not permit immediate simplification. Therefore, let us forget about the interior structure and focus on the coarsest grained structure susceptible to manipulation. It is

$$[(\underline{A}x :: h.x) \vee (\underline{A}x :: k.x)]$$

This is an expression with which we can do only one thing, to wit distribute the disjunction over the universal quantifications. In doing so, we obtain

(2) $[(\underline{A}x, u :: h.x \vee k.u)]$.

Next we can not do anything else than take into account some of the internal structure of h and k. It does however not help a thing to observe that

(3) $[h.x \equiv p.x \vee r.x]$

(4) $[k.u \equiv q.u \vee r.u]$,

because substituting this in (2) only lengthens (2) It does help however in combination with the observation that r is weakening, i.e

(5) $x \le u \Rightarrow [r.x \Rightarrow r.u]$.

This suggests that we proceed as follows

$$
\begin{array}{ll}
& (2) \\
= & \{\text{using } x \le u \vee u \le x\} \\
& [(\underline{A}x, u : x \le u : h.x \vee k.u)] \\
= & \{(3), (4)\} \\
& [(\underline{A}x, u : x \le u : p.x \vee r.x \vee q.u \vee r.u)] \\
= & \{(5)\} \\
& [(\underline{A}x, u : x \le u : p.x \vee q.u \vee r.u)] \\
= & \{\text{for } x = u, \text{ the term} \equiv \text{true, by } (0)\} \\
& [(\underline{A}x, u : x < u : p.x \vee q.u \vee r.u)] \\
= & \{\text{nothing else remains to be done} \\
& \quad \text{than substitute the full-fledged} \\
& \quad \text{expression for } r.u\} \\
& [(\underline{A}x, u : x < u : p.x \vee q.u \vee (\underline{E}y : y < u : \neg p.y \vee \neg \\
\Leftarrow & \{\text{because } x < u, \ x \text{ is in the range of } y \\
& [(\underline{A}x, u : x < u : p.x \vee q.u \vee \neg p.x \vee \neg q.x)] \\
= & \{\text{predicate calculus}\} \\
& \text{true} .
\end{array}
$$

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The crucial steps in the above calculation presumably are the transition from (1) to (2), and the observation (5) . When I encountered the theorem and tried to prove it, I discarded the transition to (2) as a starter, because it would merely complify the already complicated expression (1) by the introduction of yet another dummy. So I focused on other manipulations, but without any result.

The week before, dr. Lincoln A. Wallen had explained the flavour of Gentzen's Perfect System, which is a deduction system that can produce a proof of any "logically valid" formula. When I asked dr. Wallen to let the System work on the above theorem, the System almost immediately would generate an expression like (2). Then I produced the above proof and a fictitious heuristics. The exercise reminded thoroughly of the fact that investigating the coarse grained structure of a formula can give strong heuristic guidance. I think that that is a moral of Gentzen's Perfect System. The fact that that System presumably will identify (5) as a rabbit is just nice to know

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