Two formulae from the September '87 Top Ten

This note is written to

a) inaugurate my new MB Meisterstück, which was given to me this afternoon by Mrs. and Mr. Dijkstra.

b) to open the WF-series for scientific bagatelles that more appropriately belong in a diary and therefore never get written in my case.

c) to record two nice formulae from predicate calculus.

(ad b: Here I also celebrate the simultaneous death of a former series of notes with a similar goal. There I had imposed on myself the ridiculous constraints that for a story to be recorded, it had to be "surprising" (= smart, and often baffling by bluffing) and it had to fit on one page. The latter constraint was an open invitation towards achieving brevity by omission.)

×

The nicest of the two formulae definitely is

(0) for any bag V of predicates,
$$V \neq \emptyset \equiv [(\underline{E} X : X \in V : X \equiv (\underline{A} Y : Y \in V : Y))]$$

It was given to me by EWD. The formula is so nice because if you start looking at it and interpret it, you think it is false. Here is a proof of its correctness, by case-analysis alas.

For $V = \emptyset$ the equivalence in (0) holds.
For $V \neq \emptyset$ we derive the equivalence from

$$
\begin{array}{ll}
 & [(\underline{E}X : X \in V : X \equiv (\underline{A}Y : Y \in V : Y))] \\
= & \quad \{\text{definition of } \equiv\} \\
 & [(\underline{E}X : X \in V : (X \Rightarrow (\underline{A}Y : Y \in V : Y)) \\
 & \qquad\qquad \wedge\ (X \Leftarrow (\underline{A}Y : Y \in V : Y)))] \\
= & \quad \{\text{by the rule of instantiation the second} \\
 & \quad\ \ \text{conjunct} \equiv \text{true}\} \\
 & [(\underline{E}X : X \in V : X \Rightarrow (\underline{A}Y : Y \in V : Y))] \\
= & \quad \{\text{definition of } \Rightarrow\} \\
 & [(\underline{E}X : X \in V : \neg X \vee (\underline{A}Y : Y \in V : Y))] \\
= & \quad \{\vee \text{ distributes over } \underline{E} \text{ because } V \neq \emptyset\} \\
 & [(\underline{E}X : X \in V : \neg X) \vee (\underline{A}Y : Y \in V : Y)] \\
= & \quad \{\text{with De Morgan}\} \\
 & \text{true} \\
= & \quad \{\text{because } V \neq \emptyset\} \\
 & V \neq \emptyset
\end{array}
$$

Apology While writing the hint "definition of $\Rightarrow$" in the above proof I was aware that I had several options there, and I consciously made a choice. It was then that I realized that I had not done so in the hint "definition of $\equiv$". And it is precisely there where the case-analysis is generated. Had I chosen to rewite $P \equiv Q$ as $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ having a shape which is very "friendly" to the existential quantifier, there would have been no case-analysis at all. The apology can not end without the confession that in my first effort to show (0) the first two steps from the above proof were still combined.
(End of Apology.)

The second formula is

(1) $[(\underline{A}x :: p.x \equiv q.x) \Rightarrow ((\underline{A}x :: p.x) \equiv (\underline{A}x :: q.x))]$

which is recorded for its proof mainly. We begin to investigate its global structure with the purpose of manipulating it into a shape that is "friendly" to the omitted details. The coarsest global structure we should be interested in is $[R \Rightarrow (P \equiv Q)]$, because it is the coarsest structure that still does justice to the symmetry between p and q. R, P, and Q each being universal quantifications we should be eager to form conjunctions between them. (Disjunctions are likely to cause nested quantifications, and we deem that too complicated to begin with.) Guided by this it is quite likely that we all will find $R \wedge P \equiv R \wedge Q$ as a rewrite of $R \Rightarrow (P \equiv Q)$.

Next we manipulate --using the details-- $R \wedge P$ towards an expression which is symmetric in p and q, and then we are done:

$R \wedge P$
$=$ {definitions of R and P}
$(\underline{A}x :: p.x \equiv q.x) \wedge (\underline{A}x :: p.x)$
$=$ { joining the terms}
$(\underline{A}x :: (p.x \equiv q.x) \wedge p.x)$
$=$ {propositional calculus}
$(\underline{A}x :: p.x \wedge q.x)$

WHJ Feijen,
Department of Computer Sciences,
The University of Texas at Austin,
Austin, TX 78712 - 1188
U.S.A.

Austin,
17 & 18 October