The problem of the balanced segments

For a sequence $X(i: 0 \le i < N)$, $N \ge 0$, of 0's and 1's a balanced segment $X(i: p \le i < q)$ —of length $q-p$ — is defined as a segment such that

$$0 \le p \le q \le N \;\wedge\; (\underline{N}i: p \le i < q: X(i)=0) = (\underline{N}i: p \le i < q: X(i)=1)$$

It is requested to construct a program computing the maximum length of a balanced segment of $X(i: 0 \le i < N)$.

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Using standard strategies we develop

```
|[ n: int
; n:=0; r:=0
  {invariant P0: 0 ≤ n ≤ N ∧ r = the maximum length
       of a balanced segment of X(i: 0 ≤ i < n) }
; do n ≠ N
  → |[ l: int
     ; S
      { l = the minimum solution of the equation
             h: 0 ≤ h ∧ (X(i: h ≤ i < n+1) is balanced) }
     ; n:=n+1; r:= r max (n-l)
     ]|
  od
]|; print(r)
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The equation mentioned in the postcondition of S has at least one solution, viz. $n+1$. For the discovery of its minimum solution we rewrite the equation, using the definition of balancedness and using the abbreviation

$$D(p): \quad (\underline{N}i: 0 \le i < p: X(i)=0) - (\underline{N}i: 0 \le i < p: X(i)=1)\;,$$

into

$h:\ 0 \le h \ \wedge\ D(h) = D(n+1)$.

Our interest in the minimum solution can – in terms of the sequence D – be phrased as our interest in the "leftmost" occurrence of each D-value.
Hence, we strengthen $P0$ with $P1 \wedge P2 \wedge P3$, where

$P1:\quad d = D(n)$

$P2:\quad (\underline{A}h: 0 \le h < n+1:\ m < D(h) < M)$

$P3:\quad (\underline{A}s: m < s < M:\ c(s) = \text{the minimum solution of } h:\ 0 \le h \ \wedge\ D(h) = s)$,

so that the required value of l, mentioned in the postcondition of S, is related to c by

$(P1 \wedge P2 \wedge P3)(n+1/n) \ \Rightarrow\ l = c(d)$.

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In checking the program text below the reader is suggested to check the invariances of $P1$, $P2$, and $P3$ in succession. One hint is given for the invariance of $P3$

```
|[ n, d, m, M: int; c(i: -N ≤ i ≤ N): array of int
; n:= 0; d:= 0; m:= -1; M:= 1; c:(0) = 0 ; r:= 0
; do n ≠ N
  → if X(n) = 0 → d:= d+1 [] X(n) = 1 → d:= d-1 fi {P1(n+1/n)}
   ; if m < d ∧ d < M → skip
     [] m = d → {P1(n+1/n) ∧ P2 ∧ P3 ∧ m = d}
                c:(m) = n+1; m:= m-1 {P3(n+1/n), see Hint}
     [] d = M → c:(M) = n+1; M:= M+1
     fi
   ; n:= n+1 ; r:= r max (n - c(d))
  od
]|; print(r).
```

Hint

$$
\begin{array}{ll}
 & wp(\text{"}c{:}(m) = n+1 ;\ m := m-1\text{"},\ P3(n+1/n)) \\
= & \{[P3 \equiv P3(n+1/n)]\} \\
 & wp(\text{"}c{:}(m) = n+1 ;\ m := m-1\text{"},\ P3) \\
= & \{\text{definition of } P3 \text{, and predicate calculus}\} \\
 & P3 \wedge n+1 = \text{the minimum solution of} \\
 & \qquad\qquad h{:}\ 0 \le h \wedge D(h) = m \\
\Leftarrow & \{\text{from } P2 ,\ (\underline{A}h{:}\ 0 \le h < n+1 : D(h) \ne m)\} \\
 & P2 \wedge P3 \wedge n+1 = \text{the minimum solution of} \\
 & \qquad\qquad h{:}\ n+1 \le h \wedge D(h) = m \\
\Leftarrow & P2 \wedge P3 \wedge m = d \wedge n+1 = \text{the minimum solution} \\
 & \qquad\qquad \text{of } h{:}\ n+1 \le h \wedge D(h) = d \\
\Leftarrow & \{\text{from } P1(n+1/n) ,\ d = D(n+1)\} \\
 & P1(n+1/n) \wedge P2 \wedge P3 \wedge m = d .
\end{array}
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Quod erat demonstrandum

(End of Hint.)

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