Binary split.

Let $X(i: 0 \leq i < N)$, $N \geq 0$, be an array such that each of its elements is either red or white.

Permute the array elements such that all red elements are moved to the low end and all white elements to the high end, i.e. such that

$$(\underline{A}\, i,j : 0 \leq i < j < N : (X(i) \text{ is red}) \vee (X(j) \text{ is white}))$$

is established.

<u>Ternary split</u> (, or the problem of the Dutch National Flag).

Let $X(i: 0 \le i < N)$, $N \ge 0$, be such that each of its elements is either red or white or blue.

Permute the elements of the array X such that all red elements are low adjusted, all blue elements high adjusted, and hence all white elements in between. More precisely, if red < white < blue, permute the elements of X such that

$$(\underline{A}\, i, j: 0 \le i < j < N: \quad X(i) \le X(j))$$

is established.

Lexicographic comparison.

Let $A, B(i : 0 \leq i < N)$, $N \geq 0$, be two integer sequences and let $A < B$ mean that the A-sequence comes before the B-sequence in the lexicographic order.

Write a program to resolve the question whether $A = B$ or $A < B$ or $B < A$.

<u>Maximal variety</u> (, or Sjefke's fourth pre-course problem).

For a sequence $X(i: 0 \le i < N)$, $N \ge 1$, determine the maximum value of h such that for some $j: h \le j < N$ the subsequence $X(i: j-h \le i < j)$ contains h mutually distinct elements.

Bitonicity test

Let $X(i: 0 \leq i < N)$, $N \geq 2$, be an integer array. For $0 \leq j < N$ we define the sequence X_j as

X_j: the sequence $X(i: j \leq i < j+N)$, with indices reduced modulo N.

Write a program to solve bitone from the equation

$$\text{bitone} = (\underline{E}j :: X_j \text{ is a decreasing sequence}).$$

Klammergebirge test.

The syntactic category <K> specifies a set of character sequences. It is defined by

$$\langle K \rangle ::= \{ a \langle K \rangle b \},$$

where a and b are terminal characters.

Determine whether or not the given character sequence $X(i: 0 \leq i < 2.N)$, $N \geq 0$, belongs to the category <K>.

Long words copy.

Let CF be a character file which takes the form of a <sent>:

<sent> ::= <dot>
| <word> {<space><word>} <dot> ,
<word> ::= <letter> {<letter>} ,
and
a <letter> is a letter, a <space> is a space, and a <dot> is a dot.

Write a program that prints a copy of CF from which all words containing less than ten letters are deleted and which has the format of a <sent>.

Coincidence count

Let $X(i: 0 \le i < M)$ and $Y(j: 0 \le j < N)$ be two increasing integer sequences, $M \ge 0$ and $N \ge 0$.

Write a program to compute an integer c satisfying

$$c = (\underline{N}\, i, j: 0 \le i < M, 0 \le j < N: \; X(i) = Y(j)) .$$

Horner's scheme

Let X be an integer and $C(i: 0 \le i < N)$, $N \ge 0$, an integer array.

Write a program to compute the value of the expression

$$(\underline{S}\, i: 0 \le i < N: \ C(i) * X^i)$$

Fibonacci numbers.

The Fibonacci numbers are defined by the recurrence scheme

$$f_0 = 0, \quad f_1 = 1,$$
$$f_{k+2} = f_{k+1} + f_k \qquad \text{for all natural } k.$$

a) Given a natural N, construct a program to compute f_N

b) Show that the relations

$$f_{2k-1} = f_k^2 + f_{k-1}^2 \qquad \text{and}$$
$$f_{2k} = f_k^2 + 2 f_k f_{k-1}$$

hold for $k \geq 1$, and write another program to compute f_N for a given natural N.

Partition in F.

Let $F(i: i \geq 0)$ be an ascending sequence of positive integers. Let F also be unbounded, i.e.

$$(\underline{A}y:: (\underline{E}x: x \geq 0: F(x) > y))$$

Given a positive integer X, write a program to compute a value c satisfying

$$c = (\underline{N}i, j: 0 \leq i < j: X = (\underline{S}k: i \leq k < j: F(k))).$$

FGH.

Let F, G, H $(i : i \geq 0)$ be three ascending integer sequences and let there be at least one value occurring in all three of them.

Write a program to compute the smallest value occurring in all three sequences.

<u>Symmetric difference.</u>

Let $X(i: 0 \leq i \leq M)$ and $Y(j: 0 \leq j \leq N)$ be two ascending integer sequences such that $X(M) = Y(N) = inf.$

Write a program that prints all elements of either of the two sequences that do not occur in the other sequence.

Assortment count.

Let $X(i: 0 \le i < N)$, $N \ge 0$, be an ascending integer sequence, and let $f(x)$ be defined as

$$f(x) = (\underline{N}\, i : 0 \le i < N : X(i) = x) .$$

Write a program that computes c such that

$$c = (\underline{N}\, x : f(x) = 1)$$

is established.

Mode frequency (,or the Grigri problem).

Let $X(i: 0 \leq i < N)$, $N \geq 0$, be an ascending integer sequence, and let $f(x)$ be defined as

$$f(x) = (\underline{N} i: 0 \leq i < N: X(i) = x) .$$

Write a program to compute the maximum value of $f(x)$.

Period of a periodic sequence.

In terms of a given integer function $f(x)$ we define an infinite sequence $x_0, x_1, \ldots$:

$$x_0 = 0 \quad , \quad x_{i+1} = f(x_i) \text{ for all natural } i .$$

Let the sequence be periodic on the long run, i.e. there exists a pair (i, q): $i \geq 0$, $q \geq 1$ such that $x_i = x_{i+q}$. The least value of q among all these (i,q)-pairs yields the period of the sequence.

Write a program to compute the period of the sequence, using a fixed number of auxiliary scalar variables only.

Array element nearest to a number.

Let $A(i: 0 \le i < N)$, $N \ge 1$, be an ascending sequence of integers and let X be an integer.

Write a program that computes a value $k: 0 \le k < N$ for which $abs(A(k) - X)$ is minimal.

Array element nearest to array element.

Let $X(i: 0 \le i \le M)$ and $Y(j: 0 \le j \le N)$ be two ascending integer sequences such that $M \ge 1$, $N \ge 1$, and
$X(M-1) \ll X(M) = Y(N) \gg Y(N-1)$.

Write a program that computes a pair (i, j): $0 \le i < M \wedge 0 \le j < N$ for which $\mathrm{abs}(X(i) - Y(j))$ is minimal.

Search in an ascending matrix.

Let $A(m, n : 0 \leq m < M, 0 \leq n < N)$ be a matrix of integers such that each row is ascending (in the column index) and each column is ascending (in the row index). Let X be the value of some matrix element.

Compute a pair (i, j) of indices such that

$$X = A(i, j).$$

Museum.

Let $A, D\ (0 \le i < N)$, $N \ge 1$, be two integer sequences such that A is ascending and such that $A(i) \le D(i)$ for all i: $0 \le i < N$.
A museum is visited by N persons, numbered from 0 through $N-1$. Person i enters the museum at moment $A(i)$ and leaves it at moment $D(i)$.

Write a program that computes the total amount of time during which at least one person is inside the museum.

Cubes of odd integers.

For any pair of integers (r, k) such that $odd(r) \wedge 1 \leq r < 2^k$ an integer value x exists such that

$$odd(x) \wedge 1 \leq x < 2^k \wedge 2^k \mid (x^3 - r) .$$

(The notation "$a|b$" is short for "a divides b".)

The above theorem can be proven by designing a program that constructs such a value x.
Prove the theorem.

<u>Harmonica</u> (, or Folderol).

Let $X(i: 1 \le i \le M)$ be an increasing integer sequence, $M \ge 2$.

Write a program that computes the total number of ascending integer sequences $Y(i: 0 \le i \le M)$ satisfying the property

$$(\underline{A}\, i: 1 \le i \le M: \;\; X(i) = (Y(i-1) + Y(i))/2 \;).$$

The syntactic category C.

The syntactic category C is defined as

$$\langle C \rangle ::= \{ x \; y \; \langle C \rangle \; z \} .$$

Here x, y, and z are (terminal) characters, and the construct $\{ \dots \}$ stands for zero or more successions of the enclosed.

Given a sequence $t(i: 0 \le i < 3 \cdot N)$, $N \ge 0$, of characters such that

$$(\underline{A} i: 0 \le i < 3 \cdot N: \; t(i) = x \;\vee\; t(i) = y \;\vee\; t(i) = z) ,$$

write a program that establishes whether or not the sequence t belongs to the syntactic category C.

In honour of the little Carl Friedrich

Compute the number of ways in which a given positive integer can be written as the sum of consecutive positive integers.

H - sequences.

The set H of bit sequences is defined by the scheme

- 0 belongs to H
- if both h_0 and h_1 belong to H, so does the concatenation $1 \bullet h_0 \bullet h_1$ ($\bullet$ is a symbol to denote concatenation)
- only those sequences that can be formed by applications of the above two rules belong to H.

For a given bit sequence $h(i: 0 \leq i < 2 \cdot N+1)$, $N \geq 1$, write a program to determine whether or not the sequence $h(0) \bullet h(1) \bullet \ldots \bullet h(2 \cdot N)$ belongs to the set H.

Table for the Möbius - function.

The Möbius - function $M(n)$ is defined for each integer $n: n \geq 1$, viz.

- $M(1) = 1$;
- $M(n) = (-1)^k$ whenever n can be written as the product of exactly k different primes;
- $M(n) = 0$ in all other cases.

Generate, for given $N: N \geq 1$, the table $M(n: 1 \leq n \leq N)$.

Klaassen's integration problem.

Given a (large) positive integer N, two integers A and B such that $A \leq B$, and an integer valued step function $F(x)$ which is defined for all x: $A \leq x \leq B$.
(A stepfunction is a piecewise constant ascending function.)

Write a program that computes L such that

$$\mathrm{abs}\left(L - \int_A^B F(x)\,dx \right) < \frac{1}{N} ,$$

where it should be borne in mind that the computation of F-values could very well be expensive.
(If so desired the real-arithmetic may be thought of as presenting no additional problems.)

Simple subset-test.

Let $F(i: 0 \leq i < M)$, $M \geq 0$, be an integer sequence and let N be a natural number.

Write a program to compute f from the equation

$$f = (\underline{A} j : 0 \leq j < N : (\underline{E} i : 0 \leq i < M : F(i) = j))$$

The maximum length of a slope.

Let $F(h: 0 \le h < N)$, $N \ge 1$, be an integer sequence. A subsequence $F(h: i \le h < j)$ of F such that

$$0 \le i < j \le N$$
$$\wedge \quad F(h: i \le h < j) \text{ is ascending or descending}$$

is called "a slope of F of length $j-i$".

Write a program to compute the maximum length of any slope of F.

Some Pythagorean Triples

Given two positive integers A and D such that $D < A$, write a program to construct all triples (a, b, c) of positive integers such that

$$a < A \quad \wedge \quad a = b + D \quad \wedge \quad a^2 = b^2 + c^2 .$$

Superposition of intervals.

Let $X, Y\ (i: 0 \le i < 10)$ be two integer sequences such that $X(i) \le Y(i)$ for all $i: 0 \le i < 10$. On an initially white number line the intervals $(X(i), Y(i)): 0 \le i < 10$ are painted black.

Write a program to compute the total length of whatever looks black on the number line.

The next permutation with repetitions.

Let $X(i: 0 \le i < M)$ be a permutation of the numbers 0 through $N-1$ with repetitions, i.e. let X be such that

$$(\underline{A}\, j: 0 \le j < N: \ \underline{E}\, i: 0 \le i < M: \ X(i) = j) \quad .$$

Let X differ from the lexicographically last permutation with repetitions and let Y be its lexicographic successor.

Write a program that alters X such as to establish $X = Y$.

Subsequence frequency.

Let $X(i: 0 \le i < M)$ and $Y(j: 0 \le j < N)$ be two integer sequences, $1 \le N \le M$.
A subsequence of X of length k is what remains of X after deletion of $M-k$ elements of X.

Write a program that computes the number of subsequences of X which are equal to Y.

Rotation in situ.

Let D be a natural number and $X(i: 0 \le i < N)$ an array, $0 \le D < N$.
Define the sequence $Y(j: 0 \le j < N)$ as

$$Y(j) = X((j + D) \underline{\text{mod}}\ N) \quad \text{for all } j: 0 \le j < N .$$

Write a program to permute X -- by interchanging its elements -- so as to establish $X = Y$.

A point and a convex polygon

Let $X, Y(i : 0 \leq i < N)$, $N \geq 3$, be two integer sequences.
Let $(X(i), Y(i))$ be the Cartesian coordinates of a point P_i in the plane.
The points $P_0, P_1, \ldots, P_{N-1}$ are the successive vertices of a convex polygon.

Write a program that determines whether a grid point with coordinates (A, B) is either on or inside or outside the polygon.

Recurring subsequences.

Let $X(i: 0 \le i < N)$ be a sequence.
A subsequence $X(i: p \le i < q)$ of length $q-p$ of X, $0 \le p \le q \le N$, is a recurring subsequence of X if for some $d: d \ne 0 \wedge 0 \le p+d \wedge q+d \le N$

$$X(i: p \le i < q) = X(i: p+d \le i < q+d)$$

Write a program to compute the maximum length of any recurring subsequence of X.

The lexicographic rank of a permutation

For a permutation $X(i : 0 \le i < N)$, $N \ge 1$, of the numbers 0 through $N-1$ and a number R such that $0 \le R < N!$ we are interested in the relation

H: R = the number of permutations of 0 through $N-1$ which precede X in the lexicographic order.

a) Write a program that, given X, computes R such that H is satisfied.

b) Write a program that, given R, computes X such that H is satisfied.

The torch and the gunpowder.

Let $F(i: 0 \le i < N)$ be an integer function such that $0 \le F(i) < N$ for all $i: 0 \le i < N$, and let A and B be integer numbers such that $0 \le A < N$ and $0 \le B < N$.
There are N places, numbered from 0 through N-1.
Two men, one with a burning torch and the other one with dry gunpowder, reside at the places A and B respectively.
At gong-time a man in place x moves to place F(x).
An unlimited succession of gong-times follows.

Write a program that computes the boolean bang such that

bang = the two men are ever in the same place at the same time.

A problem on partitioning a weighted tree.

A rooted tree T has N nodes, numbered from 0 through $N-1$, $N \geq 1$; 0 is the root. Let $F(i: 1 \leq i < N)$ be such that for each node $i: 1 \leq i < N$: node $F(i)$ is the father of i in T.

Let L be a positive integer and $W(i: 0 \leq i < N)$ an array of positive integers such that

$$L \leq (\Sigma i: 0 \leq i < N: W(i)) .$$

For each node $i: 0 \leq i < N$: $W(i)$ is the weight of i in T.
The weight of a tree is the sum of the weights of its nodes.

Removal of an edge from a tree partitions the tree into two trees.

Write a program to compute the maximum number k of edge removals from T such that each of the $k+1$ resulting trees have weights which are at least L.

Netty's Diameter Problem.

On a (convex) polygon with N vertices, clockwise numbered from 0 through N-1, the distance between two vertices i and j equals D(i, j), which is the length of the clockwise path (along the circumference) from i to j. D is expressible in terms of the numbers d(i), being the length of the edge connecting vertex i with its clockwise neighbour.
A diameter of the polygon is a pair of vertices (u, v) for which $|D(u,v) - D(v,u)|$ is minimal.

Write a program to compute a diameter of the given polygon.

Wim's Diameter Problem.

The diameter of a convex polygon is the minimum value of the distance between two (distinct) parallel tangent lines

Write a program to compute the diameter of a given convex polygon (Questions about the representation of the polygon should be dealt with, but decisions about representation are preferably postponed as long as possible.)