The problem of the circular racecourse.

Given two integer arrays $p, q\ [0..N-1]$, $N \geq 1$, such that

$$(\forall i : 0 \leq i < N : \quad p[i] \geq 0 \wedge q[i] > 0), \quad \text{and}$$

$$\sum_{i=0}^{N-1} p[i] = \sum_{i=0}^{N-1} q[i] .$$

Along a circular racecourse are N pits, clockwise numbered from 0 through $N-1$. The amount of petrol available at pit i equals $p[i]$, whereas the amount of petrol needed to travel from pit i to the clockwise next pit equals $q(i)$.

Write a program to determine all pits from which a car, with an initially empty and sufficiently large tank, can start and complete the whole course in clockwise direction.

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Notational remark: For reasons of simplifying our notation we shall assume that indices submitted to the arrays p and q are to be regarded as reduced modulo N. The same is assumed of pit numbers. (End of notational remark.)

A necessary and sufficient condition for the car to round the circuit, when starting at pit k, is

$(p[k] - q[k]) \geq 0$,
so that pit $k+1$ can be reached, and

$(p[k]-q[k]) + (p[k+1]-q[k+1]) \geq 0$,
so that also pit $k+2$ can be reached, and
... , and
$(p[k]-q[k]) + (p[k+1]-q[k+1]) + \ldots + (p[k+N-1]-q[k+N-1]) \geq 0$,
so that pit $k+N$, which is pit k, can be reached as well.

We can formulate this condition a little bit more precisely by writing

$$C(k, N): \quad \left(\forall h: 0 \leq h < N: \ \sum_{i=k}^{k+h} (p[i]-q[i]) \geq 0\right).$$

Now, we could proceed by checking for each pit k whether or not $C(k, N)$ is fulfilled. Such a progression, however, would be much too wasteful, because the up to N^2 partial sums which are in the game, are mutually strongly dependent.
In order to do better, we might observe that, for instance,

$$\sum_{i=k}^{k+h} (p[i]-q[i]) = \left(\sum_{i=0}^{k+h} - \sum_{i=0}^{k-1}\right)(p[i]-q[i]),$$

which has the virtue that it reduces arbitrary partial sums to special ones, viz. those from 0 onwards.
If we introduce as an abbreviation

$$x_k = \sum_{i=0}^{k-1} (p[i]-q[i]), \qquad k \geq 0$$

then

$$C(k, N) = \left(\forall h: 0 \leq h < N: \ x_{k+h+1} \geq x_k\right).$$

Thanks to the fact that

$$x_N = \sum_{i=0}^{N-1} (p[i] - q[i]) = 0 ,$$

the condition $C(k, N)$ can be simplified drastically:

$$C(k, N): \qquad (\forall h: 0 \leq h < N: \; x_h \geq x_k) ,$$

expressing that $x_k = min\ (x_0, x_1, \ldots, x_{N-1})$.

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We can capture the desired net effect of the program to be constructed as the computation of the set Z of pit numbers such that the relation

$$R: \qquad Z = \{ k \mid 0 \leq k < N \wedge C(k, N) \}$$

is satisfied.
We propose, almost standardly so, to develop the program by sticking to the invariant

$$P: \qquad 0 \leq n \leq N \;\wedge\; Z = \{ k \mid 0 \leq k < n \wedge C(k, n) \} ,$$

strengthened, for rather obvious reasons, with

$$Q0: \qquad xx = min\ (x_0, x_1, \ldots, x_{n-1}) \qquad \text{and}$$

$$Q1: \qquad x = x_{n-1} \quad \left(= \sum_{i=0}^{n-2} (p[i] - q[i])\right)$$

respectively.

If, furthermore, the set Z is represented as

$$Z = \{ z[0], z[1], \ldots, z[l-1] \} ,$$

in which z refers to an integer array and l is an integer, the program is readily constructed. The detailed derivation of it is, by now, left as an exercise

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n:=1;  z[0]:=0;  l:=1;  xx:=0;  x:=0;
do n ≠ N →
      x:= x + p[n-1] - q[n-1];
      if x > xx → skip
      [] x = xx → z[l]:=n; l:=l+1
      [] x < xx → xx:=x;  z[0]:=n; l:=1
      fi;
      n:=n+1
od.
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Remark: The above program is such that, when under execution, the elements $p[N-1]$ and $q[N-1]$ of the arrays p and q are never inspected. Explain why this is not amazing.
(End of remark.)

Remark: One of the outcomes of the above analysis is that the condition $\sum_{i=0}^{N-1} p[i] = \sum_{i=0}^{N-1} q[i]$ is a sufficient condition for the existence of at least one pit from which a car can round the circuit. Try to prove this in a totally different way.
(End of remark.)

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Many people, students and programmers, who tried to solve this problem have made a mess of it. The main reason for this was that they could not forget about the car, the gasoline and the race circuit. In their minds they "saw" the car driving and based their thoughts on that picture. In such a world the above solution can only emerge if one is able to imagine that a car can go on with a negative amount of petrol in its tank, a mental jump which is hard to undertake!
In our ability to reason well, the choice of metaphor or nomenclature is extremely important. The wrong choice can easily lead to a complete mental blocking. There are no general rules to tell us what constitutes a good nomenclature. However, there is one rule that sometimes , and even more often, works well: collect from the problem in question the relevant mathematical properties, and that is what has been done in solving the problem of the circular race-course.

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Postscript:
Almost all of the heuristics involved in the presented solution is suggested by J.L.A. van de Snepscheut and is much more elegant than the old story.
(End of postscript.)