Set of programming exercises (Santa Cruz, august 1979).

1) Usually the Fibonacci numbers are defined by the recurrence scheme

$f_0 = 0, \quad f_1 = 1,$

$f_k = f_{k-1} + f_{k-2}$ for all $k: k \geq 2$.

a) Write a program to compute, for a given integer $N: N \geq 0$, the value of f_N .

b) Show that the Fibonacci numbers satisfy the relations

$f_{2k-1} = f_k^2 + f_{k-1}^2$ and $f_{2k} = f_k^2 + 2f_k f_{k-1}$ for all $k: k \geq 1$,

and, again, write a program to compute f_N , for a given integer $N: N \geq 0$.

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2) Let x be a given integer, and let $c(0..N)$ be a given array of integers, $N \geq 0$.

Write a program to compute the value of the polynomial

$c(0)x^N + c(1)x^{N-1} + \ldots + c(N-1)x + c(N)$.

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3) Given three integer-valued functions $f(i)$, $g(i)$ and $h(i)$, each defined for all integers $i: i \geq 0$ and enjoying the properties

- $f(0) \leq f(1) \leq f(2) \leq \ldots$
- $g(0) \leq g(1) \leq g(2) \leq \ldots$
- $h(0) \leq h(1) \leq h(2) \leq \ldots$
- there is at least one value that occurs in all three sequences f, g and h.

Write a program to compute the minimal value that occurs in all three sequences f, g and h.

4) For any pair of integers (r,k) satisfying: odd(r) <u>and</u> $1 \leqslant r < 2^k$, there exists an integer value x such that:

odd(x) <u>and</u> $1 \leqslant x < 2^k$ <u>and</u> $2^k \mid (x^3 - r)$.

(The notation " $a \mid b$ " is short for " a divides b ".)

The above theorem can be proven by designing a program that constructs such a value x .
Prove the theorem.

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5) Given an integer M: $M \geqslant 0$, and an integer array S(0..N-1): $N \geqslant 0$.

Write a program to determine whether or not

($\underline{A}$m: $0 \leqslant m < M$: $\underline{E}$n: $0 \leqslant n < N$: S(n) = m)

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6) Let x be an integer, and d(0..N-1) an integer array satisfying:

$N \geqslant 1$ <u>and</u> $d(0) \leqslant d(1) \leqslant \ldots \leqslant d(N-1)$.

Write a program to compute an array element d(k) as close to x as possible, i.e. a value of k: $0 \leqslant k < N$ for which $|d(k) - x|$ is minimal.

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7) Construct the largest number of which the decimal representation can be formed by a rearrangement of the digits of the decimal representation of a given integer N: $N \geqslant 1$.

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8) Given an integer array $r(0..N-1)$, construct a sequence $c(0), c(1), ..., c(N)$ such that for all (real) values of x :

$$c(0)x^N + c(1)x^{N-1} + ... + c(N) = (x-r_0)(x-r_1)...(x-r_{N-1})$$

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9) Let $x_1, x_2, ...$ be an infinite sequence of numbers, defined -- in terms of an integer-valued function $f(x)$ -- as

$$x_1 = 1,$$
$$x_{i+1} = f(x_i) \quad \text{for all } i: i \geqslant 1.$$

The function is such that the sequence $x_1, x_2, ...$ is periodic in the long run, i.e. there exists a pair (i,q): $i,q \geqslant 1,1$ for which $x_i = x_{i+q}$. The smallest value of q among all these pairs (i,q) yields the period of the sequence $x_1, x_2, ...$.

Write a program to compute the period of the sequence. It is required that the amount of storage used is independent of the sequence $x_1, x_2, ...$.

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10) Consider the set H of bitsequences that can be formed according to the following rules:

a. 0 belongs to H
b. if both h_1 and h_2 belong to H , so does the concatenation $1 . h_1 . h_2$ (. is a symbol to denote concatenation)
c. only those sequences that can be formed by application of the rules a. and b. , belong to H .

Given an array $h(0..2N)$ of bits, $N \geqslant 1$, write a program to determine whether or not the sequence $h(0) . h(1) . ... . h(2N)$ belongs to the set H .

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11) Given two integers A and D, $1 \leqslant D < A$.

Write a program to generate all integer triples (a,b,c) satisfying:

- $a, b, c \geqslant 1, 1, 1$ <u>and</u> $a \leqslant A$
- $a - b = D$ <u>and</u> $a^2 - b^2 = c^2$.

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12) a) Let $f(i)$ be an integer-valued function, defined for all integers i: $i \geqslant 1$, and let f have the property

$$0 \leqslant f(1) < f(2) < f(3) < \ldots .$$

It is requested to compute, for a given integer A: $A \geqslant 1$, the total number of different pairs (p,q) such that

- $1 \leqslant p \leqslant q$
- $A = f(p) + f(p+1) + \ldots + f(q)$.

b) Write an entirely different program for the special case where $f(i) = i$.

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13) Let $f(i)$ be an integer-valued function, defined for all integers i: $0 \leqslant i \leqslant N$, $N \geqslant 0$.
A subsequence $f(i), f(i+1), \ldots, f(j)$ of the sequence $f(0), f(1), \ldots, f(N)$ is said to be a <u>slope</u> of f if and only if

- $0 \leqslant i \leqslant j \leqslant N$
- the sequence $f(i), f(i+1), \ldots, f(j)$ is either monotonically nondecreasing or monotonically nonincreasing.

The <u>length</u> of a slope $f(i), f(i+1), \ldots, f(j)$ of f equals $j-i+1$.

Write a program for the computation of the maximum length of a slope of f.

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14) Given two integer arrays p, $q(0..N-1)$, $N \geqslant 1$. The arrays are such that

- $p(i) \geqslant 0$, $q(i) > 0$ for all $i: i \geqslant 0$
- $p(0) + p(1) + \ldots + p(N-1) = q(0) + q(1) + \ldots + q(N-1)$.

Along a circular racecourse there are N pits, clockwise numbered from 0 through N-1 . The amount of petrol present at pit i equals p(i) , whereas the amount of petrol needed to reach from pit i the clockwise next pit equals q(i) .

Write a program to determine all pits from which a car, with an initially empty and sufficiently large tank, can complete the whole racecourse in the clockwise direction.

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15) Given an integer $N: N \geqslant 1$.

a) Write a program that can, if so desired, generate all integer sequences $(p_1, p_2, \ldots)$ such that

- $p_i \geqslant 1$ for all $i: i \geqslant 1$
- $p_1 + p_2 + \ldots = N$

b) Write a program that can, if still desired, generate all integer sequences $(p_1, p_2, \ldots)$ such that

- $1 \leqslant p_1 \leqslant p_2 \leqslant \ldots$
- $p_1 + p_2 + \ldots = N$.

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16) Given two integer arrays a, $b(1..N)$, $N > 1$, with the property

$a(i) \leqslant N$ <u>and</u> $b(i) \leqslant N$ <u>and</u> $a(i) \neq b(i)$ for all $i: 1 \leqslant i \leqslant N$.

The arcs of a directed graph, consisting of the N nodes 1 through N , are given via the following rules:

a. for i: $1 \leq i \leq N$, there is an arc from node i to node
   a) $a(i)$ if and only if $a(i) > i$
   b) $b(i)$ if and only if $b(i) > i$

b. all arcs that belong to the graph, belong to it on account of rule a. .

Write a program to compute the total number of different paths leading from node 1 to node N .

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17) Let $m(1..M)$ be an integer array such that

$$M \geq 2 \text{ and } m(1) < m(2) < \ldots < m(M) .$$

Write a program to compute the total number of integer sequences $v_0, v_1, \ldots, v_M$ satisfying the properties

- $v_0 \leq v_1 \leq \ldots \leq v_M$
- $(v_{i-1} + v_i)/2 = m(i)$ for all i: $1 \leq i \leq M$.

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18) The syntactic category C is defined as follows:

$$\langle C \rangle ::= x\ y\ \{\langle C \rangle\}\ z .$$

Here, x, y and z are (terminal) characters, and the construct $\{\ldots\}$ stands for zero or more successions of the enclosed.

Given an array $T(1..3000)$ of characters such that $T(i) = x$ <u>or</u> $T(i) = y$ <u>or</u> $T(i) = z$ for all i: $1 \leq i \leq 3000$, write a program to determine whether or not the character sequence $T(1), T(2), \ldots, T(3000)$ is of the syntactic category C .

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19) Given two integer arrays a, d(1..N), $N \geq 1$. The arrays are such that $a(i) \leqslant d(i)$ for all $i: 1 \leqslant i \leqslant N$.
A museum is visited by N persons, numbered from 1 through N. Person i enters the museum at moment a(i), and leaves it again at moment d(i).

Write a program to compute the total amount of time during which at least one person is inside the museum.

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20) The Möbius-function M(n) is defined for each integer $n: n \geqslant 1$ by the following rules:

- M(1) = 1
- $M(n) = (-1)^k$ if n can be written as the product of exactly k different primes
- M(n) = 0 in all other cases.

Write a program to compose, for a given integer $N: N \geqslant 1$, a table for M(n), for n running from 1 to N.

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21) Given two integer arrays f(0..F), g(0..G), $F \geqslant 1$, $G \geqslant 1$, such that

- $f(F) = g(G)$, $f(F-1) < f(F)$, $g(G-1) < g(G)$
- the f-sequence is monotonically nondecreasing, i.e. $f(0) \leqslant \ldots \leqslant f(F)$
- the g-sequence is monotonically nondecreasing, i.e. $g(0) \leqslant \ldots \leqslant g(G)$.

Write a program to extract from the two sequences all elements the value of which does occur in the one sequence but not in the other: f(i) belongs to the extract if and only if $0 \leqslant i \leqslant F$ and ($\underline{A}$j: $0 \leqslant j \leqslant G$: $g(j) \neq f(i)$), and similar for an element g(j).

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22) Given an integer array $x(0..N-1)$ and an integer p: $0 \leq p < N$. It is requested to write a program that rotates the contents of the array cyclically over p positions to the left, i.e.
if the initial value of $x(i)$ is denoted by a_i , then the program should establish

$$x(i) = a_{(i+p)\underline{mod}\ N} \quad \text{for all } i: 0 \leq i < N .$$

The program has to be such that the array is only changed by operations x: swap(i,j), which exchanges the element-values in the positions i and j .

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23) An integer function $f(x,y)$ of integer arguments x and y enjoys the following properties:

- $f(x,y) > x$ for all x, y
- if $y_1 < y_2$ then $f(x,y_1) < f(x,y_2)$ for all x, y_1, y_2.

A subset V of the natural numbers is defined as follows:

a. 1 belongs to V

b. if both x and y belong to V , so does $f(x,y)$

c. all elements belonging to V belong to V on account of a. and b. .

Write a program to compute the thousand smallest elements of V .

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24) Given an integer array $s(0..N-1)$, $N \geq 1$.

A so-called P-sequence , of length k , is an integer sequence $p_0, p_1, \ldots, p_{k-1}$ such that

- $k \geq 1$
- $0 \leq p_i < N$ for all $i: 0 \leq i < k$
- $p_{i-1} < p_i \leq s(p_{i-1})$ for all $i: 0 < i < k$.

Write a program to compute the maximal length of a P-sequence.

25) Given two integer arrays x, y(0..46) . The pair (x(i),y(i)) represents the Cartesian coordinates of a point P_i in a plane $(0 \leq i < 47)$. The arrays are such that all 47 points are different.

A robot walks from P_0 to P_1, from P_1 to P_2, ..., from P_{45} to P_{46}, and finally from P_{46} to P_0. It does so in obedience to the following rules:

- it shall start from P_0 whilst looking towards P_1 ;
- it shall always walk into the direction in which it is looking;
- it shall only alter its direction of looking in the points P_i , namely by performing a clockwise rotation of alpha: $0^\circ \leq$ alpha $< 360^\circ$;
- it shall end in P_0 whilst looking towards P_1 .

As a consequence of this walk the robot has made a clockwise rotation which is a multiple of 360°.

Write a program to compute this multiple, under the additional restriction that all expressions and variables occurring in the program have to be of integer or of boolean type.

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26) Compute, for a given integer $N: N \geq 1$, the period of the decimal expansion of $1/N$.

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27) Let k and N be two given integers such that $0 < k \leq N$.
A sequence $(e_1, \ldots, e_{2N})$ is said to have property E if and only if:

- $e_i = \pm 1$ <u>and</u> $0 \leq e_1 + e_2 + \ldots + e_i \leq k$ for all $i: 1 \leq i \leq 2N$
- $e_1 + e_2 + \ldots + e_{2N} = 0$.

Let $(e_1, \ldots, e_{2N})$ and $(f_1, \ldots f_{2N})$ be two different sequences having property E , and let m be the smallest index, $1 \leq m \leq 2N$, for which $e_m \neq f_m$, then the sequence $(e_1, \ldots, e_{2N})$ is said to precede the sequence $(f_1, \ldots, f_{2N})$ in the <u>alphabetic</u> order if and only if $e_m < f_m$.

For a given array s(1..2N) such that the sequence (s(1),...,s(2N)) has property E and is not the alphabetically last one, write a program to transform (s(1),...,s(2N)) into the alphabetically immediate successor with property E .

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28) Given an integer t0 and two integer arrays a, b(1..N), $N \geq 1$. The arrays are such that

- $a(1) < a(2) < \ldots < a(N)$
- $b(i) > 0$ for all $i: 1 \leq i \leq N$.

A post-office with four equivalent service desks is used by N customers, numbered from 1 through N . The post-office opens at moment t0 . Customer i arrives at moment a(i) , and has a service request of b(i) time units. Service is granted in the order in which the customers arrive. The behaviour of the desk officers is such that they are never idle when a customer is waiting for service. The post-office closes as soon as all N customers have been served.

Write a program to compute the total amount of time during which at least one customer is waiting for service.

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29) Given two integers a, b and three integer arrays x, y1, y2(1..N) , such that

- $N \geq 1$
- $a < x(1) < x(2) < \ldots < x(N) < b$
- $y1(i) < y2(i)$ for all $i: 1 \leq i \leq N$.

In a landscape are N hedges. Hedge i extends between the points with coordinates (x(i),y1(i)) and (x(i),y2(i)) , $1 \leq i \leq N$.
A nature-lover wants to walk from the point with coordinates (a,0) to the point with coordinates (b,0) , thereby not pounding hedges.

Write a program to compute the shortest road for the nature-lover to travel.

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30) Given three integers X, Y and N, satisfying $1 \leq X < N$, $1 \leq Y < N$ and $(X,Y) \neq (N-1,1)$.
On the set of all integer pairs (a,b): $1 \leq a < N$, $1 \leq b < N$ we define the total ordering as imposed by the relation "$\prec$" :

$$(a,b) \prec (c,d) \quad \text{if and only if}$$
$$\frac{a}{b} < \frac{c}{d} \;\underline{\text{or}}\; \left(\frac{a}{b} = \frac{c}{d} \;\underline{\text{and}}\; a > c \right).$$

The pair (c,d) is called the immediate successor of the pair (a,b) if and only if

$$(a,b) \prec (c,d) \;\underline{\text{and}}$$
$$(\text{for all other pairs } (e,f)\text{: } (e,f) \prec (a,b) \;\underline{\text{or}}\; (c,d) \prec (e,f)) .$$

Write a program to compute the immediate successor of (X,Y) .

(End of set of programming exercises
(Santa Cruz, august 1979).)

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July 29, 1979
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