IMO 2007, Problem 4

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Problem Statement

The original problem statement reads:

> **Problem 4.** In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Solution

Let's start with a diagram (see Figure 1) presenting the givens.

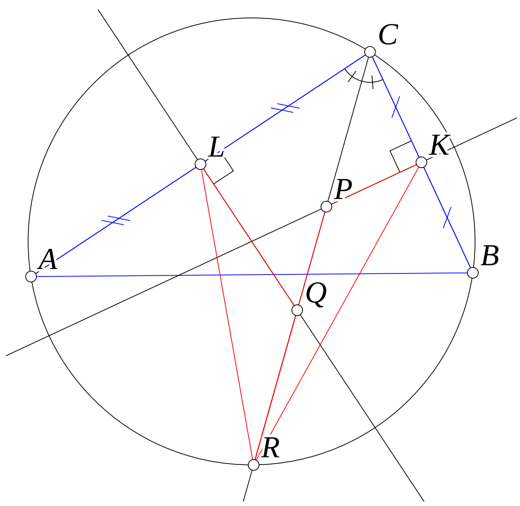

Figure 1: Triangle ABC (blue), and triangles RPK and RQL (red)

We calculate

$$
\begin{array}{ll}
 & \text{area } RPK \;=\; \text{area } RQL \\
\equiv & \quad \{\ KS \text{ is the altitude at } K;\ LT \text{ is the altitude at } L\ \} \\
 & \frac{1}{2}|RP| \cdot |KS| \;=\; \frac{1}{2}|RQ| \cdot |LT| \\
\equiv & \quad \{\ \text{algebra, the lengths involved are nonzero}\ \} \\
 & |RP| : |RQ| \;=\; |LT| : |KS| \\
\equiv & \quad \{\ \text{triangles } CPK \text{ and } CQL \text{ are similar, having two angles in common}\ \} \\
 & |RP| : |RQ| \;=\; |CQ| : |CP| \\
\equiv & \quad \{\ P \text{ and } Q \text{ lie on } RC\text{; rewrite } |CQ| \text{ and } |CP|\ \} \\
 & |RP| : |RQ| \;=\; |CR| - |RQ| : |CR| - |RP| \\
\equiv & \quad \{\ \text{introduce names } a, b, c = |RP|, |RQ|, |CR|\ \} \\
 & a : b \;=\; c - b : c - a \\
\equiv & \quad \{\ \text{algebra; the values involved are nonzero}\ \} \\
 & a(c-a) = b(c-b) \\
\equiv & \quad \{\ \text{algebra}\ \} \\
 & a(c-a) - b(c-b) = 0 \\
\equiv & \quad \{\ \text{algebra}\ \} \\
 & (a-b)c - a^2 + b^2 = 0 \\
\equiv & \quad \{\ \text{algebra}\ \} \\
 & (a-b)(c-(a+b)) = 0 \\
\equiv & \quad \{\ \text{algebra}\ \} \\
 & a = b \ \vee\ c = a + b \\
\Leftarrow & \quad \{\ a \text{ and } b \text{ are independent, } a, b, c = |RP|, |RQ|, |CR|\ \} \\
 & |CR| \;=\; |RP| + |RQ| \\
\equiv & \quad \{\ |CR| = |RP| + |CP|\ \} \\
 & |RQ| = |CP|
\end{array}
$$

Therefore, it suffices to prove $|RQ| = |CP|$. For that purpose, we introduce the line through the circumcenter M perpendicular to angle bisector CR (red in Figure 2). Now, reflect triangle ABC in this line to obtain triangle $A'B'R$ (dashed blue in Figure 2).

Triangles ABC and $A'B'R$ have the same circumcircle, since the relection line passes through the circumcenter. Sides AC and $B'R$ are parallel, because of equal angles with CR. Similarly, sides BC and $A'R$ are parallel. The perpendicular bisector of $B'R$ passes through the circumcenter M and is perpendicular to AC. Hence, the perpendicular bisectors of AC and $B'R$ are the same. Consequently, Q is the reflection of P and, thus, $|RQ| = |CP|$.

Q.E.D.

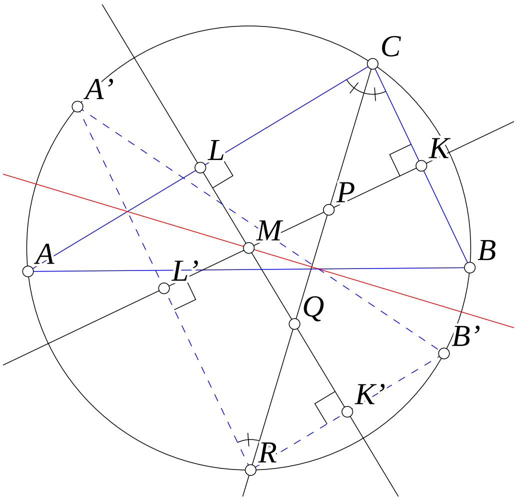

Figure 2: Triangle ABC (blue), reflection line (red), reflected triangle $A'B'R$ (dashed blue)