IMO 2007, Problem 1

Tom Verhoeff

4 August 2007

Introduction

I will present a calculational solution to Problem 1 of the 48th International Mathematical Olympiad (IMO) held in July 2007 in Hanoi, Vietnam [3]. This is the first of six problems at IMO 2007. On each of the two competition days, the contestants were given three problems to be solved in four and a half hours.

Problem Statement

The original problem statement reads:

Problem 1. Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \leq i \leq n)$ define

$$d_i = \max\{a_j : 1 \leq j \leq i\} - \min\{a_j : i \leq j \leq n\}$$

and let

$$d = \max\{d_i : 1 \leq i \leq n\}.$$

(a) Prove that, for any real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$,

$$\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ such that equality holds in $(*)$.

Solution

It surprises me a bit that the problem statement mentions no restrictions on n, for example that it is a nonnegative integer. The case $n = 0$ is uninteresting (or ill-defined, if you prefer). So, let us assume $1 \leq n$.

I will not constantly repeat the range restriction on indices. Indices range over the interval $\{1, 2, \ldots, n\}$. For maximum and minimum I use the notation $\uparrow$ and $\downarrow$.

The notation that I use is based on that of Edsger W. Dijkstra [1], who wrote a large collection of (mostly technical) essays [2], many of them with inspiring solutions to nice problems.

The problem statement can now be disentangled as follows:

Given is a sequence a of n real numbers. For each i define

$$\begin{array}{rcl} m.i & = & (\downarrow j : i \leq j : a.j) \\ M.i & = & (\uparrow j : j \leq i : a.j) \\ d.i & = & M.i - m.i \\ d & = & (\uparrow i :: d.i) \ . \end{array}$$

(a) Prove that, for any ascending sequence x of n real numbers,

$$(\uparrow i :: |x.i - a.i|) \ \geq \ \frac{d}{2} \ . \qquad (1)$$

(b) Show that there is an ascending sequence x of n real numbers with

$$(\uparrow i :: |x.i - a.i|) \ = \ \frac{d}{2} \ . \qquad (2)$$

Note the different ranges in the definitions of $m.i$ and $M.i$. For increasing i, the range of $m.i$ *decreases* (in terms of set containment), and the range of $M.i$ *increases*. Hence, both $m.i$ and $M.i$ are ascending in i:

$$\begin{array}{rcl} m.i & \leq & m.j \\ M.i & \leq & M.j \end{array}$$

for $i \leq j$. It is also worthwhile to observe that for all i

$$m.i \ \leq \ a.i \ \leq \ M.i \ . \qquad (3)$$

This yields $d.i \geq 0$ and, hence, also $d \geq 0$.

The goal of part (a) is to prove (1), which can be restated as

$$(\exists\, i :: |x.i - a.i| \geq d/2) \tag{4}$$

The definition of d involves several quantifications. Let's analyze it:

$$\begin{array}{ll} & d = (\uparrow\ i :: d.i) \\ \Rightarrow & \quad \{ \text{ property of } \uparrow\text{, guided by } \exists\text{-shape of goal (4) } \} \\ & (\exists\, i :: d = d.i) \\ \equiv & \quad \{ \text{ definition of } d.i \ \} \\ & (\exists\, i :: d = M.i - m.i) \\ \equiv & \quad \{ \text{ definitions of } M.i \text{ and } m.i\text{, using a fresh dummy for } m.i \ \} \\ & (\exists\, i :: d = (\uparrow\ j : j \leq i : a.j) - (\downarrow\ k : i \leq k : a.k)) \\ \Rightarrow & \quad \{ \text{ property of } \uparrow \text{ and } \downarrow\text{, guided by } \exists\text{-shape of goal (4) } \} \\ & (\exists\, i, j, k : j \leq i \leq k : d = a.j - a.k))) \end{array}$$

Now, take j and k with $j \leq k$ such that

$$a.j - a.k \quad = \quad d\ . \tag{5}$$

Note that $x.j \leq x.k$ and thus

$$x.k - x.j \quad \geq \quad 0\ . \tag{6}$$

Adding (5) and (6) yields

$$a.j - x.j + x.k - a.k \quad \geq \quad d\ . \tag{7}$$

Consequently (generalized pigeon-hole principle, or converse of monotonicity of addition),

$$a.j - x.j \geq d/2 \quad \vee \quad x.k - a.k \geq d/2\ .$$

Using $d \geq 0$, this establishes (4) and thereby settles part (a) of the problem.

Part (b)

Once we have part (a), the goal of part (b) can be weakened to

> Show that there is an ascending sequence x of n real numbers with
>
> $$(\uparrow\ i :: |x.i - a.i|) \quad \leq \quad \frac{d}{2}\ . \tag{8}$$

Equation (8) can be rephrased as

$$(\forall\, i :: |x.i - a.i| \le d/2) \ . \qquad (9)$$

By definition of d we have

$$(\forall\, i :: M.i - m.i \le d) \qquad (10)$$

In view of (3) — $m.i \le a.i \le M.i$ — it seems sweetly reasonable to take

$$x.i \quad = \quad \frac{M.i + m.i}{2} \ ,$$

that is, take $x.i$ as midpoint of $m.i$ and $M.i$. It remains to ascertain that this sequence x is ascending and that it satisfies (9).

Ascendingness follows immediately from the ascendingness of m and M. And (9) follows from (3) and (10). This is intuitively obvious, but here is a calculation:

$$\begin{array}{ll}
 & |a.i - x.i| \\
= & \quad \{ \text{ definition of } x.i \ \} \\
 & |a.i - (M.i + m.i)/2| \\
= & \quad \{ \text{ rearrange terms } \} \\
 & |(a.i - M.i)/2 + (a.i - m.i)/2| \\
\le & \quad \{ \text{ triangle inequality } \} \\
 & |(a.i - M.i)/2| + |(a.i - m.i)/2| \\
= & \quad \{ \text{ (3): } m.i \le a.i \le M.i \ \} \\
 & (M.i - a.i)/2 + (a.i - m.i)/2 \\
= & \quad \{ \text{ algebra } \} \\
 & (M.i - m.i)/2 \\
\le & \quad \{ \text{ (10) } \} \\
 & d/2
\end{array}$$

Q.E.D.

Conclusion

I must confess that initially I drew some diagrams, only to find out that they did not really help me in giving a *rigorous* and *elegant* proof. In hindsight, studying the case $n = 2$ actually suffices for this problem.

The definition of d could have been simplified as follows, but it is not necessary to discover this explicitly:

$$
\begin{array}{ll}
 & d \\
= & \quad \{ \text{ definition of } d \text{ } \} \\
 & (\uparrow\ i :: d.i) \\
= & \quad \{ \text{ definition of } d.i \text{ } \} \\
 & (\uparrow\ i :: M.i - m.i) \\
= & \quad \{ \text{ definition of } M.i \text{ and } m.i \text{, using a fresh dummy for } m.i \text{ } \} \\
 & (\uparrow\ i :: (\uparrow\ j : j \le i : a.j) - (\downarrow\ k : i \le k : a.k)) \\
= & \quad \{\ -(\downarrow k :: E.k) = (\uparrow\ k :: -E.k)\ \} \\
 & (\uparrow\ i :: (\uparrow\ j : j \le i : a.j) + (\uparrow\ k : i \le k : -a.k))) \\
= & \quad \{ \text{ distribute } + \text{ over } \uparrow \text{ (nonempty range, twice) } \} \\
 & (\uparrow\ i :: (\uparrow\ j,k : j \le i \le k : a.j - a.k))) \\
= & \quad \{ \text{ change order of } \uparrow \text{ quantifcations } \} \\
 & (\uparrow\ j,k : j \le k : (\uparrow\ i : j \le i \le k : a.j - a.k))) \\
= & \quad \{\ a.j - a.k \text{ does not depend on } i \text{, the range for } i \text{ is nonempty } \} \\
 & (\uparrow\ j,k : j \le k : a.j - a.k)))
\end{array}
$$

There are various alternative definitions for sequence x in part (b), such as

$$
\begin{aligned}
x.i &= M.i - d/2 \\
x.i &= m.i + d/2\ .
\end{aligned}
$$

In summary, the two "key" properties used in my solution are:

$$
\begin{aligned}
b + c \ge d &\quad\Rightarrow\quad b \ge \frac{d}{2}\ \vee\ c \ge \frac{d}{2} \\
b \le a \le c &\quad\Rightarrow\quad \left| a - \frac{b+c}{2} \right| \le \frac{c-b}{2}\ .
\end{aligned}
$$

References

[1] Edsger W. Dijkstra, 1930–2002.
Web: `http://en.wikipedia.org/wiki/Edsger_Dijkstra`

[2] E. W. Dijkstra Archive, University of Texas at Austin.
Web: `http://www.cs.utexas.edu/users/EWD/`,

[3] 48th International Mathematical Olympiad, 19–31 July 2007, Vietnam.
Web: `http://www.imo-official.org/year_country_r.asp?year=2007`