McCarthy's 91-function: a sequel to EWD845

McCarthy's *91-function* is the function g defined recursively as follows:

(g0) $g{\cdot}x = x-10 \quad , \; 100<x$

(g1) $g{\cdot}x = g{\cdot}(g{\cdot}(x+11)) \; , \; x\leqslant 100$

The problem is to prove that g equals the function f defined by:

(f) $f{\cdot}x = 91 \ \mathbf{max}\ (x-10)$

In this note, we shall do so by mathematical induction on the domains of the functions, where $100<x$ constitutes the "base" and where the "step" is in the direction of decreasing values. Here we go.

For $x : 100<x$ we derive:

$g{\cdot}x$
$= \{ \text{(g0)} \}$
$x-10$
$= \{ 100<x \text{ , hence: } 91 \leqslant x-10 \}$
$91 \ \mathbf{max}\ (x-10)$
$= \{ \text{(f)} \}$
$f{\cdot}x$.

For $x : x\leqslant 100$ we derive:

$g{\cdot}x$
$= \{ \text{(g1)} \}$
$g{\cdot}(g{\cdot}(x+11))$
$= \{ x<x+11 : \text{induction hypothesis} \}$
$g{\cdot}(f{\cdot}(x+11))$
$= \{ \text{(f)} \}$
$g{\cdot}(91 \ \mathbf{max}\ (x+1))$
$= \{ x < 91 \ \mathbf{max}\ (x+1) : \text{induction hypothesis} \}$
$f{\cdot}(91 \ \mathbf{max}\ (x+1))$
$= \{ \text{(f)} \}$

$$
\begin{aligned}
& 91 \;\mathbf{max}\; ((91 \;\mathbf{max}\; (x+1)) - 10) \\
= & \{ \text{ distribution of } - \text{ over } \mathbf{max} \} \\
& 91 \;\mathbf{max}\; 81 \;\mathbf{max}\; (x-9) \\
= & \{ 91 \;\mathbf{max}\; 81 = 91 \} \\
& 91 \;\mathbf{max}\; (x-9) \\
= & \{ x \leqslant 100 \text{ , hence: } x-9 \leqslant 91 \text{ and } x-10 \leqslant 91 \} \\
& 91 \;\mathbf{max}\; (x-10) \\
= & \{ \text{(f)} \} \\
& f{\cdot}x \quad . \qquad\qquad\qquad \text{Q.E.D.}
\end{aligned}
$$

Browsing through old EWD's I encountered EWD845 [0] . In order to exercise my current ability in proving properties of recursively defined functions, I decided to prove the equality of f and g myself. The amount of case analysis involved could be reduced even further by using the definition of f as given above, instead of the more conventional:

$$
\begin{aligned}
f{\cdot}x &= x-10 \;,\; 100 < x \\
f{\cdot}x &= 91 \quad\;\;\; , \; x \leqslant 100
\end{aligned}
$$

Notice that the above proof is of the type "nothing else you can do". I like it so much that -- in spite of the last sentence of EWD845 -- I neither can nor wish to resist the temptation to record it: the above demonstration I consider neither lengthy nor boring.

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reference

[0] Edsger W. Dijkstra: "McCarthy's 91-function: an unfortunate paradigm" (EWD845).

(**postscriptum** on the next page →→→)

Postscriptum

Having read an earlier version of this note, Kees Hemerik directed my attention to a paper [1] published in 1973 containing essentially the same proof -- attributed to R.M. Burstall -- as the one given here (Example 22 in [1]). So, what are we talking about? Apparently, we do not know our literature sufficiently well....

Eindhoven, 1987.9.29
Rob Hoogerwoord

reference

[1] Z. Manna, S. Ness, J. Vuillemin: "Inductive methods for proving properties of programs", CACM 16(8) (1973), pp 491-502.