A Useful Classification

0 Several Kinds of Relations

Throughout this little essay we study relations from a given set B to a given set V, so we study subsets of the cartesion product $B \times V$. For R such a relation we write bRv as an abbreviation of $(b,v) \in R$, as usual.

For R such a relation we define four properties that R may or may not have:

(0) "R is L-total" $\Leftrightarrow$ $(\forall b : b \in B : (\exists v : v \in V : bRv))$

(1) "R is L-functional" $\Leftrightarrow$ $(\forall b,u,v : b \in B \wedge u,v \in V : bRu \wedge bRv \Rightarrow u = v)$

(2) "R is R-total" $\Leftrightarrow$ $(\forall v : v \in V : (\exists b : b \in B : bRv))$

(3) "R is R-functional" $\Leftrightarrow$ $(\forall b,c,v : b,c \in B \wedge v \in V : bRv \wedge cRv \Rightarrow b = c)$

Note that "R is L-total" expresses that every element in B is related (by R) to *at least one* element in V, whereas "R is L-functional" expresses that every element in B is related to *at most one* element in V. Hence, together – in conjunction – they express that every element in B is related to *exactly one* element in V.

Similarly, "R is R-total" expresses that every element in V is related (by R) to *at least one* element in B, whereas "R is R-functional" expresses that every element in V is related to *at most one* element in B. So, together they express that every element in V is related to *exactly one* element in B.

Also note the symmetry between (0) and (1) on the one hand and (2) and (3) on the other hand: the two pairs are transformed into one another under relation transposition. That is, for every relation R we have:

"R is L-total" $\Leftrightarrow$ "R^{T} is R-total" , and:

"R is L-functional" $\Leftrightarrow$ "R^{T} is R-functional" .

As far as I know, Netty van Gasteren, in her PhD-thesis, was the first to emphasize the importance of these four properties (although she used different names).

1 Several Kinds of Functions

Using the terminology introduced above we can now classify several kinds of functions, in the following way. Again, R is a relation from B to V.

"R is a (partial) function (from B to V)" $\Leftrightarrow$
"R is L-functional" ,

"R is a (total) function" $\Leftrightarrow$
"R is L-total" $\wedge$ "R is L-functional" ,

"R is a surjective function" $\Leftrightarrow$
"R is L-total" $\wedge$ "R is L-functional" $\wedge$ "R is R-total" ,

"R is an injective function" $\Leftrightarrow$
"R is L-total" $\wedge$ "R is L-functional" $\wedge$ "R is R-functional" ,

"R is a bijection" $\Leftrightarrow$
"R is L-total" $\wedge$ "R is L-functional" $\wedge$ "R is R-total" $\wedge$ "R is R-functional" .

In particular this classification clarifies the relation between functions and their inverses (if they exist). We have, for instance, that relation R is a bijection if and only if both R and R^{T} are (total) functions, which then are each other's inverses, as we will show.

2 Applications

To demonstrate the usefulness of the above concepts we formulate and prove two lemmata.

Lemma 0: Every bijection R satisfies: $R\,;R^{\mathrm{T}} = I_B$, where " ; " denotes relation composition and where I_B denotes the identity relation (on $B\times B$). Similarly, $R^{\mathrm{T}}\,;R = I_V$.

proof: We prove $R\,;R^{\mathrm{T}} = I_B$ element-wise, that is, we prove $(\forall b,c : b,c\in B : b(R\,;R^{\mathrm{T}})c \Leftrightarrow b=c)$, as follows, for any $b,c\in B$:

$b\,(R\,;R^{\mathrm{T}})\,c$

$\Leftrightarrow$ { definition of ; }

$(\exists v : v\in V : bRv \wedge vR^{\mathrm{T}}c)$

$\Leftrightarrow$ { definition of T }

$(\exists v : v\in V : bRv \wedge cRv)$

$\Leftrightarrow$ { R is R-functional: (3) }

$(\exists v : v\in V : bRv \wedge cRv \wedge b=c)$

$\Leftrightarrow$ { Leibniz, to simplify }

$(\exists v : v\in V : bRv \wedge b=c)$

$\Leftrightarrow$ { $\wedge$ distributes over $\exists$ }

$(\exists v : v\in V : bRv) \wedge b=c$

$\Leftrightarrow$ { R is L-total: (0) }

$b=c$.

Notice that, in this proof, we only have used that R is R-functional and L-total. The other two properties, that R is L-functional and R-total, are needed for the proof of the symmetric counterpart $R^{\mathrm{T}}\,;R = I_V$.

□

The next lemma actually is an exercise in our course named “Discrete Structures”; the students tend to consider this exercise a difficult one.

Lemma 1 : For R a relation from B to V and S a relation from V to B we have: if $R\,;S = I_B$ and $S\,;R = I_V$ then both R and S are bijections.

proof: The problem is entirely symmetric in R and S, in that it is invariant under the substitution $B,V,R,S := V,B,S,R$. Hence, it suffices to prove that R is a bijection. As we have seen, this means that R has the four properties (0) through (3). The problem also is symmetric in that it is invariant under relation transposition. Hence, as we have seen, it suffices to prove that R is L-total and L-functional only. To be able to do so, however, we must rephrase our premisses $R\,;S = I_B$ and $S\,;R = I_V$ in a way that does not involve relation composition anymore, because properties (0) through (3) do not contain compositions.

Rewriting $R\,;S = I_B$ as the conjunction of $R\,;S \subseteq I_B$ and $I_B \subseteq R\,;S$ and $S\,;R = I_V$ as the conjunction of $S\,;R \subseteq I_V$ and $I_V \subseteq S\,;R$ respectively, and eliminating the compositions and identities −by applying their definitions− our premisses boil down to the following four given properties:

(4) $(\forall b,c : b,c{\in}B : (\exists v : v{\in}V : bRv \wedge vSc) \Rightarrow b{=}c)$

(5) $(\forall b : b{\in}B : (\exists v : v{\in}V : bRv \wedge vSb))$

(6) $(\forall u,v : u,v{\in}V : (\exists b : b{\in}B : uSb \wedge bRv) \Rightarrow u{=}v)$

(7) $(\forall v : v{\in}V : (\exists b : b{\in}B : vSb \wedge bRv))$

Now we prove that R is L-total, as follows:

R is L-total
$\Leftrightarrow$ { (0) }
$(\forall b : b{\in}B : (\exists v : v{\in}V : bRv))$
$\Leftarrow$ { strengthening, in view of (5) }
$(\forall b : b{\in}B : (\exists v : v{\in}V : bRv \wedge vSb))$
$\Leftrightarrow$ { (5) }
true ,

and, hence, by the aforementioned symmetries, we conclude that S is R-total too. Now, we prove that R is L-functional, as follows, using definition (1), for all $b \in B$ and $u,v \in V$:

$bRu \wedge bRv$
$\Leftrightarrow$ { S is R-total }
$(\exists w : w{\in}V : wSb) \wedge bRu \wedge bRv$
$\Leftrightarrow$ { $\wedge$ distributes over $\exists$ }

$$
\begin{array}{ll}
 & (\exists w : w{\in}V : wSb \wedge bRu) \;\wedge\; bRv \\
\Leftrightarrow & \quad \{\ (6)\,,\ \text{with}\ u,v := w,u\ \} \\
 & (\exists w : w{\in}V : wSb \wedge bRu \wedge w{=}u) \;\wedge\; bRv \\
\Leftrightarrow & \quad \{\ \text{one-point rule}\ \} \\
 & uSb \;\wedge\; bRu \;\wedge\; bRv \\
\Rightarrow & \quad \{\ \text{weakening}\ \} \\
 & uSb \;\wedge\; bRv \\
\Rightarrow & \quad \{\ \exists\text{-introduction}\ \} \\
 & (\exists b : b{\in}B : uSb \wedge bRv) \\
\Rightarrow & \quad \{\ (6)\ \} \\
 & u = v \quad ,
\end{array}
$$

as required. Notice that a step like the first one in this calculation is unavoidable: the proof obligation is about R only, so we must introduce, one way or another, relation S into the game, in order to be able to use the premisses (4) through (7), all of which involve both R and S.

□

3 A more abstract and more algebraic approach

In the proof of Lemma 1 I have observed that the premisses of the lemma contain relation composition, whereas the proof obligations, amounting to (0) through (3), do not contain relation composition. Therefore, so I decided, we must eliminate relation composition from the premisses, which gave rise to formulae (4) through (7).

When writing down this observation, however, it suddenly dawned upon me that there is an alternative approach: instead of *eliminating* relation composition from the premisses we might as well *introduce* it into formulae (0) through (3), thus lifting the whole discussion to a more abstract, element-free, level.

To be able to do so, I need a general (logical) property which I will refer to either as "$\exists$-elimination" or as "$\exists$-introduction". In a natural-deduction style of logic this rule, when applied from left to right, is usually called "$\exists$-elimination". Note, however, that the rule actually expresses an equivalence which, therefore, may be applied from right to left as well, as a way of "$\exists$-introduction".

$\exists$-elimination: For any predicate $P(x)$, possibly containing x as a free variable, and for any predicate Q, in which x does *not* occur as free variable, we have:

$$(\exists x :: P(x)) \Rightarrow Q \quad \Leftrightarrow \quad (\forall x :: P(x) \Rightarrow Q) \quad .$$

proof: By straightforward calculation.

□

Now we can translate the classification from Section 0 into relational form, as follows:

$$
\begin{array}{ll}
 & R \text{ is L-total} \\
\Leftrightarrow & \quad \{ \ (0) \ \} \\
 & (\forall b : b \in B : (\exists v : v \in V : b R v) \ ) \\
\Leftrightarrow & \quad \{ \text{ idempotence of } \wedge \text{ and transposition } \} \\
 & (\forall b : b \in B : (\exists v : v \in V : b R v \wedge v R^{\mathrm{T}} b) \ ) \\
\Leftrightarrow & \quad \{ \text{ definition of } ; \ \} \\
 & (\forall b : b \in B : b \, (R ; R^{\mathrm{T}}) \, b \ ) \\
\Leftrightarrow & \quad \{ \text{ one-point rule } \} \\
 & (\forall b, c : b, c \in B : \ b = c \ \Rightarrow \ b \, (R ; R^{\mathrm{T}}) \, c \ ) \\
\Leftrightarrow & \quad \{ \text{ definition of } I_B \ \} \\
 & (\forall b, c : b, c \in B : \ b \, (I_B) \, c \ \Rightarrow \ b \, (R ; R^{\mathrm{T}}) \, c \ ) \\
\Leftrightarrow & \quad \{ \text{ definition of } \subseteq \ \} \\
 & I_B \subseteq R ; R^{\mathrm{T}} \quad .
\end{array}
$$

So, relation R is L-total if and only if $I_B \subseteq R ; R^{\mathrm{T}}$, and this may be taken as an alternative definition. Notice, however, that the second step in the above derivation − introduction of R^{T} − is rather arbitrary. This is a true design decision, but not the only possibility. Instead, for example, we could also have rewritten bRv to $bRv \wedge v \top b$, where $\top$ denotes the maximal relation, that is the whole set $V \times B$. This gives rise to yet another alternative definition: R is L-total if and only if $I_B \subseteq R ; \top$. As a matter of fact, there are good reasons to retain both variants, and performing the same exercises with R-totality we obtain:

(8) “R is L-total” $\Leftrightarrow$ $I_B \subseteq R ; R^{\mathrm{T}}$

(9) “R is L-total” $\Leftrightarrow$ $I_B \subseteq R ; \top$

(10) “R is R-total” $\Leftrightarrow$ $I_V \subseteq R^{\mathrm{T}} ; R$

(11) “R is R-total” $\Leftrightarrow$ $I_V \subseteq \top ; R$

Note that (9) is weaker than (8) and that (11) is weaker than (10): when we need to *prove* totality the weaker forms are to be preferred, but when we *use* totality the stronger forms are more attractive.

* * *

As to functionality we proceed as follows:

$$
\begin{array}{ll}
 & R \text{ is L-functional} \\
\Leftrightarrow & \quad \{ \ (1) \ \}
\end{array}
$$

$(\forall b,u,v : b{\in}B \wedge u,v{\in}V : bRu \wedge bRv \Rightarrow u=v\,)$

$\Leftrightarrow$ { dummy b does not occur in $u=v$: nesting }

$(\forall u,v : u,v{\in}V : (\forall b : b{\in}B : bRu \wedge bRv \Rightarrow u=v\,)\,)$

$\Leftrightarrow$ { $\exists$-introduction }

$(\forall u,v : u,v{\in}V : (\exists b : b{\in}B : bRu \wedge bRv\,) \Rightarrow u=v\,)$

$\Leftrightarrow$ { transposition }

$(\forall u,v : u,v{\in}V : (\exists b : b{\in}B : uR^{\mathrm{T}}b \wedge bRv\,) \Rightarrow u=v\,)$

$\Leftrightarrow$ { definition of ; and I_V }

$(\forall u,v : u,v{\in}V : u\,(R^{\mathrm{T}}\,;R)\,v \Rightarrow u\,(I_V\,)\,v\,)$

$\Leftrightarrow$ { definition of $\subseteq$ }

$R^{\mathrm{T}}\,;R \subseteq I_V$.

R-functionality can, of course, be rewritten in exactly the same way. Thus, we obtain:

(12) "R is L-functional" $\Leftrightarrow$ $R^{\mathrm{T}}\,;R \subseteq I_V$

(13) "R is R-functional" $\Leftrightarrow$ $R\,;R^{\mathrm{T}} \subseteq I_B$

* * *

Now let us see how the two lemmata from the previous section can be proved in terms of the above definitions.

Lemma 0 : Every bijection R satisfies $R\,;R^{\mathrm{T}} = I_B$ and $R^{\mathrm{T}}\,;R = I_V$.

proof: This is rather trivial now:

$R\,;R^{\mathrm{T}} = I_B$

$\Leftrightarrow$ { set equality via mutual inclusion }

$R\,;R^{\mathrm{T}} \subseteq I_B \;\wedge\; I_B \subseteq R\,;R^{\mathrm{T}}$

$\Leftrightarrow$ { R is a bijection, so R is R-functional, (13) , and R is L-total, (8) }

true .

□

Lemma 1 : For R a relation from B to V and S a relation from V to B we have: if $R\,;S = I_B$ and $S\,;R = I_V$ then both R and S are bijections.

proof: As we have observed already in the previous proof, it suffices to prove that R is L-total and L-functional; obviously, here we use the weaker form of totality:

R is L-total

$\Leftrightarrow$ { (9) }

$I_B \subseteq R\,;\top$

$\Leftrightarrow$ { $R\,;S = I_B$ }

$R\,;S \subseteq R\,;\top$

$\Leftarrow$ { ; is monotonic }

$S \subseteq \top$

$\Leftrightarrow$ { definition of $\top$ }

true .

Notice that use of the weaker definition, (9) , really helps here: this leads to $S \subseteq \top$, which is easy, whereas the stronger definition would have given rise to $S \subseteq R^{\mathrm{T}}$, which requires more work.

Furthermore:

R is L-functional

$\Leftrightarrow$ { (12) }

$R^{\mathrm{T}}\,;R \subseteq I_V$

$\Leftrightarrow$ { $S\,;R = I_V$ }

$R^{\mathrm{T}}\,;R \subseteq S\,;R$

$\Leftarrow$ { ; is monotonic }

$R^{\mathrm{T}} \subseteq S$

$\Leftrightarrow$ { transposition }

$R \subseteq S^{\mathrm{T}}$

$\Leftrightarrow$ { I_V is identity of ; }

$R \subseteq S^{\mathrm{T}}\,;I_V$

$\Leftrightarrow$ { $S\,;R = I_V$ }

$R \subseteq S^{\mathrm{T}}\,;S\,;R$

$\Leftrightarrow$ { I_B is identity of ; }

$I_B\,;R \subseteq S^{\mathrm{T}}\,;S\,;R$

$\Leftarrow$ { ; is monotonic }

$I_B \subseteq S^{\mathrm{T}}\,;S$

$\Leftrightarrow$ { S is R-total: (10) , with $B, V, R := V, B, S$ }

true .

From a heuristic point of view, the step labelled "transposition" is the most difficult one in this proof. It is needed to prepare for what follows, were we need to obtain I_B to the left of a set inclusion. Honesty forces me to admit here that I could only do this because I already knew that in the element-wise proof (in the previous section) I have used that S is R-total and that, therefore, it is most likely needed here as well.

□

It is somewhat amazing that, once we have formulated definitions (8) through (13) we have used no other properties of relation composition than that it has an identity element, that it is associative –which we have used implicitly–, and that it is monotonic (with respect to $\subseteq$).

4 Afterthoughts

More generally, the notion *exactly one* is an awkward one, mathematically speaking. The best way to treat it is to view it as the conjunction of the two notions *at least one* and *at most one*, which can be treated in relative isolation. In my experience, the notion *at least one* –non-emptyness– really differs from the notion *at most one* –uniqueness–.

* * *

The purely relational characterisations –formulae (8) through (13) – in terms of relation composition really are useful. They show that totality and functionality actually are faces of the same coin, and they allow for much more concise proofs. Relational algebra, with composition as an important operator, still is undervalued!

The transition even renders Lemma 0 completely trivial. Lemma 1, on the other hand, is not trivial at all; in this respect I can appreciate that our students find this a difficult exercise.

Eindhoven, 14 october 2010

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