$\exists^*$-elimination revisited

0 $\exists^*$-elimination

In their textbook *"Logical Reasoning: A First Course"* the authors, Rob Nederpelt and Fairouz Kamareddine – in what follows referred to as "N&K" – , present a natural-deduction style of "logical derivations", in which so-called *flags* are used as an educational tool to clearly visualize the scopes of assumptions and bound variables.

This style of reasoning involves introduction and elimination rules for each of the logical constructors. In particular, existential quantification initially is defined in terms of universal quantification, by means of De Morgan's rule. Subsequently, however, additional rules for existential quantification are presented, called $\exists^*$-introduction and $\exists^*$-elimination, which have more practical value.

This note is about a problem I have with $\exists^*$-elimination, as presented in the textbook, and how I propose to resolve it. The problem is as follows.

In a situation where we "have" (assumed or proved, one way or another) a proposition of the shape $\exists_x [\, P(x) : Q(x) \,]$ and, using this proposition, we wish to prove some proposition R, the reasoning with $\exists^*$-elimination, according to N&K, is as follows:

(0) $\exists_x [\, P(x) : Q(x) \,]$

{ $\exists^*$-elim on (0) }

(1) Pick an x with $P(x)$ and $Q(x)$

⋮

proof of R

⋮

(9) R

In step (1) a variable x is introduced which is assumed to satisfy $P(x)$ and $Q(x)$. The idea now is that in the subsequent "proof of R" both $P(x)$ and $Q(x)$ may be used. The result is a correct proof of R; this is justified by the presence of proposition (0). No additional assumptions on x may be made: any x is as good as any other, only its existence is relevant.

So far so good, but I take the point of view that, essentially, there is no such thing as a *free* variable, because eventually all variables are *bound* somewhere. A variable only becomes free when the context in which it is bound is ignored, but that context exists nevertheless! In addition, every bound variable should have a clearly delineated *scope*.

In the above derivation, for instance, variable x is introduced and bound in line (1), but what is its scope? As said already, variable x may be used, in the form of propositions $P(x)$ and $Q(x)$, in the proof of R, but once this proof is complete, variable x is not needed anymore. So, somewhere around line (9), the scope of x should end. My problem now is that, in the way N&K render $\exists^*$-elimination, this scope is *not* clearly delineated. Mind you that x really is a bound variable here; for example, the rule of renaming bound variables is applicable: we might equally well have written "Pick a y with $P(y)$ and $Q(y)$" and then formulate the "proof of R" in terms of $P(y)$ and $Q(y)$.

So, how do we resolve this? To remain in style with N&K, where every (other) scope is clearly delineated by a flag, I would like to use a flag in this situation as well. In the above reasoning scheme, this flag would fly on line (1) and its pole would preferably end around line (9). And, I would even like to remain as close as possible to the style of, say, the flags used in $\forall$-introduction – which, actually, is the only other way to introduce (local) variables – .

In order to develop a proper and correct rendering of $\exists^*$-elimination, we must analyze first what $\exists^*$-elimination really means. It so happens that the above reference to $\forall$-introduction is not coincidental at all: as a matter of fact, as we will see, $\exists^*$-elimination actually *is* $\forall$-introduction in disguise!

In addition, years of teaching on this subject shows that students often experience $\exists^*$-elimination as difficult and that they often don't know how to write down proofs involving it properly. The lack of clarity with respect to the role and the scope of the variable may be the cause of this. By demonstrating that $\exists^*$-elimination essentially boils down to $\forall$-introduction brings the students back onto familiar grounds and, thus, reduces the confusion.

1 Intermezzo: A crucial distribution property

In the chapters on calculations with quantified expressions N&K have failed to include the following, rather crucial, rule that expresses that *disjunction distributes over universal quantification*. More precisely, for predicates $P(x)$ and $Q(x)$, and for any predicate R in which x does not occur as a free variable, we have:

$$(0) \qquad \forall_x [\, P(x) : Q(x) \,] \vee R \;\stackrel{val}{=}\; \forall_x [\, P(x) : Q(x) \vee R \,] \quad .$$

Notice that the requirement that x does not occur freely in R can be easily met by a suitable renaming of the bound variable in the quantified expressions.

Not surprisingly, there also is a dual rule that expresses that conjunction distributes over existential quantification. By means of De Morgan's rule the latter can be proved from the former, but there is no way[0] in which (0) can be proved from the calculational rules in N&K's text. So, if we wish to have (0) as a rule in the repertoire it must be included as a basic rule: that is why I call it crucial.

One may well wonder, then, whether rule (0) is correct. Fortunately it is, because it can be proved by means of derivations in the natural deduction style. This requires a ping-pong argument, one for each direction of the equivalence; otherwise, however, these two proofs are simple and straightforward – in which some renaming of bound variables has been performed – :

[0] Except for *finite* domains, where Mathematical Induction can be used.

proof of “ $\overset{val}{\models}$ ”:

(0) $\forall_x [\, P(x) : Q(x) \,] \vee R$

(1) **var** $y : P(y)$

(2) $\neg R$

{ $\vee$-elim on (0) and (2) }

(3) $\forall_x [\, P(x) : Q(x) \,]$

{ $\forall$-elim on (3) and (1) }

(4) $Q(y)$

{ $\vee$-intro on (4) and (2) }

(5) $Q(y) \vee R$

{ $\forall$-intro on (1) and (5) }

(6) $\forall_y [\, P(y) : Q(y) \vee R \,]$

proof of “ $\overset{val}{=\!\!|}$ ”:

(0) $\forall_x [\, P(x) : Q(x) \vee R \,]$

(1) $\neg R$

(2) **var** $y : P(y)$

{ $\forall$-elim on (0) and (2) }

(3) $Q(y) \vee R$

{ $\vee$-elim on (3) and (1) }

(4) $Q(y)$

{ $\forall$-intro on (2) and (4) }

(5) $\forall_y [\, P(y) : Q(y) \,]$

{ $\vee$-intro on (5) and (1) }

(6) $\forall_y [\, P(y) : Q(y) \,] \vee R$

Aside: It is somewhat surprising that the two proof styles seem to require different sets of basic rules. As already stated, the distribution rule must be included as a basic rule in the calculational repertoire whereas it can be proved by means of deductions. On the other hand, the basic deduction rules for $\forall$-introduction and $\forall$-elimination can be proved by means of calculations.

□

2 $\exists^*$-elimination calculationally

Now we are ready to study $\exists^*$-elimination in a calculational setting. The idea is simple: in a context where we "have" $\exists_x[\,P(x):Q(x)\,]$ we can prove a proposition R by proving the (weaker):

(1) $\exists_x[\,P(x):Q(x)\,] \;\Rightarrow\; R$.

This is sufficient because now modus ponens – $\Rightarrow$-elimination – does the job. To get rid of the $\exists$ in proposition (1) we calculate as follows:

$$
\begin{array}{ll}
 & \exists_x[\,P(x):Q(x)\,] \Rightarrow R \\
= & \quad \{\text{ implication }\} \\
 & \neg\exists_x[\,P(x):Q(x)\,] \vee R \\
= & \quad \{\text{ de Morgan }\} \\
 & \forall_x[\,P(x):\neg Q(x)\,] \vee R \\
= & \quad \{\ \vee \text{ distributes over } \forall\ \} \\
 & \forall_x[\,P(x):\neg Q(x) \vee R\,] \\
= & \quad \{\text{ implication }\} \\
 & \forall_x[\,P(x):Q(x) \Rightarrow R\,] \ .
\end{array}
$$

Thus we obtain the following lemma. I am inclined to consider this the essence of $\exists^*$-elimination.

$\exists^*$-elim lemma:

$$\exists_x[\,P(x):Q(x)\,] \Rightarrow R \quad \stackrel{val}{==} \quad \forall_x[\,P(x):Q(x) \Rightarrow R\,]$$

□

3 $\exists^*$-elimination by natural deduction

Using the above $\exists^*$-elim lemma I now know how to construct a proof for a proposition R in the natural deduction style, that depends on $\exists_x[\,P(x):Q(x)\,]$:

(0) $\exists_x [\, P(x) : Q(x) \,]$

(1) **var** $x : P(x)$

(2) $Q(x)$

⋮

proof of R

⋮

(10) R

{ $\Rightarrow$-intro on (2) and (10) }

(11) $Q(x) \Rightarrow R$

{ $\forall$-intro on (1) and (11) }

(12) $\forall_x [\, P(x) : Q(x) \Rightarrow R \,]$

{ $\exists^*$-elim lemma }

(13) $\exists_x [\, P(x) : Q(x) \,] \Rightarrow R$

{ $\Rightarrow$-elim on (0) and (13) }

(14) R .

Although this is, as yet, somewhat elaborate, this is $\exists^*$-elimination in full glory, and variable x used here indeed is the variable needed for a $\forall$-introduction. Notice that in this proof the scope of x is clearly delineated by the outer flag.

4 Cleaning up: keeping things manageable

The above proof represents the pattern of reasoning for all kinds of $\exists^*$-elimination; hence, its elaborateness is awkward. In particular, the last steps and the intermediate propositions (12) and (13) will always be the same. So, for the sake of efficiency, let us clean up this mess by introducing suitable abbreviations.

Firstly, the two flags can be combined into a single flag because, by means of "Domain Weakening", we have that $\forall_x [\, P(x) : Q(x) \Rightarrow R \,]$ is equivalent to $\forall_x [\, P(x) \wedge Q(x) : R \,]$; this can be exploited as follows:

(0) $\exists_x [\, P(x) : Q(x) \,]$

(1) | **var** $x : P(x) \wedge Q(x)$

⋮

proof of R

⋮

(9) R

{ $\forall$-intro on (1) and (9) }

(10) $\forall_x [\, P(x) \wedge Q(x) : R \,]$

{ domain weakening and $\exists^*$-elim lemma }

(11) $\exists_x [\, P(x) : Q(x) \,] \Rightarrow R$

{ $\Rightarrow$-elim on (0) and (11) }

(12) R .

Formally speaking, inside the "proof of R", one or two appeals to $\wedge$-elimination are needed in order to obtain $P(x)$ and $Q(x)$ in isolation from P(x) $\wedge$ Q(x), but whether one wishes to do so explicitly or implicitly I consider a minor concern here. In the same vein as in N&K's text we also might render the flag as follows to separate $P(x)$ and $Q(x)$ right away:

(0) $\exists_x [\, P(x) : Q(x) \,]$

(1) | **var** $x :$ $P(x)$ and $Q(x)$

⋮

proof of R

⋮

(9) R

Although the reduction from two flags to one is an improvement already, we can abbreviate the pattern even further, by leaving out the intermediate propositions (10) and (11) altogether. To bridge the gap between R at the bottom of the flag pole and the final conclusion R outside the scope of the flag, I propose to use the justification "$\exists^*$-elimination". Notice that this step now takes care of the $\forall$-introduction, the appeal to the $\exists^*$-elim lemma, and the subsequent $\Rightarrow$-elimination, in one fell swoop:

(0) $\exists_x [\, P(x) : Q(x) \,]$

(1) | **var** $x : P(x)$ and $Q(x)$

$\vdots$

proof of R

$\vdots$

(9) R

{ $\exists^*$-elim on (0), (1), and (9) }

(10) R .

Notice that proposition R occurs twice now, on lines (9) and (10); this duplication is unavoidable: firstly, R appears, *within* the scope of x, as the conclusion of the "proof of R"; secondly, this R must be taken *outside* this scope, as the final conclusion of the whole proof. This duplication of R is the (modest) price we pay for the explicit delineation of the scope of variable x. It really is necessary too: in the class room, for example, I have tried to clarify things by sticking to the N&K format –see Section 0– but by drawing a flag around the "Pick a ... " construct; this didn't work because I didn't know where to end the flag pole: *before* line (9) or *after* line (9)? Neither is correct! Not surprisingly, because in N&K's format the *single* occurrence of R plays a *double* role.

Also notice that, contrary to N&K's text, the justicification $\exists^*$-elimination now appears *after* the "proof of R", and the whole proof is introduced by the flag in line (1). This is not as bad as one might may be tempted to believe. After all, natural-deduction proofs preferably are constructed from bottom to top, with the goals to be proved leading the way. In our case, assuming that proposition (0) somehow already is present, and with proposition R at line (10) as our goal to be proved, this is the initial configuration:

(0) $\exists_x [\, P(x) : Q(x) \,]$

$\vdots$

???

$\vdots$

(10) R .

In this situation it is quite a reasonable design decision to use $\exists^*$-elimination to weaken the proof obligation. Phrased differently, the decision to employ $\exists^*$-elimination is about the first step one takes, in order to create additional context for the proof of R, so as to weaken the proof obligation. This process is very much the same as the way $\forall$-introduction is used; in view of the preceding story this should not come as a surprise, of course. As a result of such this decision the configuration becomes:

(0) $\exists_x [\, P(x) : Q(x) \,]$

(1) | **var** $x : P(x)$ and $Q(x)$

| $\vdots$

| ???

| $\vdots$

(9) | R

{ $\exists^*$-elim on (0), (1), and (9) }

(10) R .

This is the way in which I think proofs with $\exists^*$-elimination should be rendered. An additional advantage, of course, is that the somewhat strange announcement "Pick a ... " has disappeared: what, actually, is that supposed to mean? Well, my story shows that the answer is remarkably simple: it is just the binding of a local variable, with a clearly visible scope, just as in the case of $\forall$-introduction.

5 Epilogue

In their textbook N&K conclude Section 15.4 – on $\exists^*$-introduction and $\exists^*$-elimination – with the following remark:

> Why is the $\exists^*$-elim rule a 'good' rule now? We have defended it *intuitively* [sic] , but this is different from having an absolute certainty of the correctness of the rule.
>
> It suffices here to say that it can indeed be *proven exclusively with the earlier given rules*, that the method of the $\exists^*$-elim rule is correct. This proof is quite involved and hence we leave it out.

The proof that the rule of $\exists^*$-elimination is correct is not "involved" at all! As I have shown, the crux of the matter is the $\exists^*$-elim lemma, and once we have formulated it life becomes easy. The very first proof in Section 3 is a completely straightforward application of this lemma, and its correctness is beyond any doubt.

All that remains to be done is to clean up the presentation by means of the introduction of a suitable abbreviation: the transition from this elaborate version to the more concise, final one, mainly is a notational issue.

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