Time to wake up

0 A nice exercise

The following exercise was brought to my attention some weeks ago. It seems to be taken from “Mathematical Miniatures”, by Svetsoslav Savchev and Titu Andreescu (The Mathematical Association of America, 2003.):

> During a certain lecture, each of the students in the hall fell asleep exactly once. For each pair of students, there was some moment when both were sleeping simultaneously. Prove that, at some moment, all students were sleeping simultaneously.

This exercise admits a nice calculational solution, provided one is familiar with the following, quite elementary, mathematics.

1 Preliminaries

The following mathematical properties are well-known in some circles but less so in others, which is why I summarize them here.

For the *maximum* of two numbers we use a binary (infix) operator $\uparrow$, and for the *minimum* of two numbers we use a binary (infix) operator $\downarrow$; that is, the larger and the smaller of two numbers x and y are written as:

$$x{\uparrow}y \text{ and } x{\downarrow}y \quad \text{, respectively} \quad .$$

These operators are idempotent, symmetric –commutative–, and associative, and they have the following properties:

(0) $\quad x{\uparrow}y \le z \;\equiv\; x \le z \wedge y \le z \quad$, for all x,y,z .

(1) $\quad z \le x{\downarrow}y \;\equiv\; x \le z \wedge y \le z \quad$, for all x,y,z .

As a matter of fact, relations (0) and (1) actually *define* $\uparrow$ and $\downarrow$; that is, all known properties of $\uparrow$ and $\downarrow$ can be derived from them. In what follows we will call (0) “the $\uparrow/\le$-connection” and we will call (1) “the $\downarrow/\le$-connection”.

Both connections admit straightforward generalizations to arbitrary finite collections of numbers, thus –where dummy p is assumed to range over a finite domain, identifying a collection of numbers x_p–:

$$(\mathsf{max}\, p :: x_p) \le z \;\equiv\; (\forall p :: x_p \le z) \quad \text{, and:}$$

$$z \le (\mathsf{min}\, p :: x_p) \;\equiv\; (\forall p :: z \le x_p) \quad .$$

These, too, we call “the $\uparrow/\le$-connection” and “the $\downarrow/\le$-connection”, respectively.

2 A little interval calculus

We consider closed intervals of real (or rational, or integer) numbers. An interval $[b..x]$ is non-empty if and only if $b \le x$. In what follows all intervals under consideration are assumed to be non-empty.

The *intersection* of two (non-empty) intervals $[b..x]$ and $[c..y]$ is empty if (and only if) one interval is "completely to the left of" the other; that is, if:

$$x < c \ \vee \ y < b \quad .$$

Now we can calculate a formal expression for two intervals "having a *non-empty* intersection":

$$
\begin{array}{ll}
 & \text{"the intersection of } [b..x] \text{ and } [c..y] \text{ is non-empty"} \\
\equiv & \quad \{ \text{ above definition of empty intersection } \} \\
 & \neg (x < c \ \vee \ y < b) \\
\equiv & \quad \{ \text{ de Morgan and a property of } < \ \} \\
 & c \le x \wedge b \le y \\
\equiv & \quad \{ \text{ both intervals are non-empty } \} \\
 & b \le x \wedge c \le x \wedge b \le y \wedge c \le y \\
\equiv & \quad \{ \uparrow / \le \text{-connection (twice) } \} \\
 & b \uparrow c \le x \ \wedge \ b \uparrow c \le y \\
\equiv & \quad \{ \downarrow / \le \text{-connection } \} \\
 & b \uparrow c \le x \downarrow y \quad .
\end{array}
$$

In what follows we will use both the intermediate result:

(2) $\quad c \le x \wedge b \le y \quad ,$

and the final result:

(3) $\quad b \uparrow c \le x \downarrow y \quad ,$

as characterizations of non-emptyness. Formula (3) generalizes easily to any finite number of intervals having a non-empty intersection too.

3 The sleepy students

Let dummy p range over the (finite) collection of students. Let b_p be the (unique) moment student p falls asleep, and let x_p be the moment student p wakes up again. If student p sleeps until the end of the lecture then x_p is the moment the lecture ends. Because it is given that every student falls asleep (once) during the lecture, so before the lecture ends, the interval $[b_p..x_p]$ is non-empty, for every p.

That for every two students a moment exists when both were sleeping simultaneously means that every two sleep intervals have a non-empty intersection. Because every sleep interval itself is non-empty this is even true if the two students (and their intervals) are the *same*. That at some moment *all* students were sleeping simultaneously means that *all* sleep intervals have a non-empty intersection. That the former proposition implies the latter we prove now as follows, by calculation:

$(\forall p, q :: \text{“the intersection of } [b_p \,..\, x_p] \text{ and } [b_q \,..\, x_q] \text{ is non-empty”})$

$\equiv$ { characterization (2) of non-empty intersections }

$(\forall p, q :: b_q \le x_p \wedge b_p \le x_q)$

$\equiv$ { term splitting }

$(\forall p, q :: b_q \le x_p) \;\wedge\; (\forall p, q :: b_p \le x_q)$

$\equiv$ { dummy renaming $p, q := q, p$ in the left term; $\wedge$ is idempotent }

$(\forall p, q :: b_p \le x_q)$

$\equiv$ { nesting dummies }

$(\forall q :: (\forall p :: b_p \le x_q))$

$\equiv$ { $\uparrow / \le$ -connection }

$(\forall q :: (\mathsf{max}\, p :: b_p) \le x_q)$

$\equiv$ { $\downarrow / \le$ -connection }

$(\mathsf{max}\, p :: b_p) \le (\mathsf{min}\, q :: x_q)$

$\equiv$ { (generalized) characterization (3) of non-empty intersections }

“the intersection of all intervals $[b_p \,..\, x_p]$ is non-empty” .

Apart from being calculational and free from unnecessary case distinctions, this proof is, for someone who is sufficiently familiar with the $\uparrow/\le$- and $\downarrow/\le$-connections, entirely of the kind only-one-thing-you-can-do.

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