Range/Term Trading: a generalization

0 Dummy Transformations

Let X and V be sets and let φ be a surjective function of type $X \to V$. Let $\oplus$ be a binary operator that is idempotent, commutative, associative, and that has an identity element $\mathbf{0}$. With this operator we can associate a quantor which we write here as $\mathbf{Q}$.

Any such quantor admits the following rule for *dummy transformation*:

(0) $$(\mathbf{Q}u : u \in V : F \cdot u) \quad = \quad (\mathbf{Q}x : x \in X : F \cdot (\varphi \cdot x)) \quad .$$

The surjectivity of φ guarantees that all elements of V are "reached from" X (at least once) and the idempotence of $\oplus$ makes it harmless if some elements of V are "reached" more than once. (For non-idempotent operators function φ must, therefore, be *injective* as well.)

Now suppose that the range of quantification is the union of two sets V and W, say, and suppose that we have two sets X and Y, and two surjective functions φ and ψ, of types $X \to V$ and $Y \to W$ respectively. Then, we can calculate as follows:

$$
\begin{array}{ll}
 & (\mathbf{Q}u : u \in V \cup W : F \cdot u) \\
= & \quad \{ \text{ range split } \} \\
 & (\mathbf{Q}u : u \in V : F \cdot u) \oplus (\mathbf{Q}u : u \in W : F \cdot u) \\
= & \quad \{ \text{ dummy transformation (twice) } \} \\
 & (\mathbf{Q}x : x \in X : F \cdot (\varphi \cdot x)) \oplus (\mathbf{Q}y : y \in Y : F \cdot (\psi \cdot y)) \quad .
\end{array}
$$

1 Range/term trading revisited

The above is quite elementary but how do we perform these calculations if the range of quantification contains an additional predicate $P \cdot u$, say, as a conjunct? That is, how do we calculate with formulae of the shape:

(1) $$(\mathbf{Q}u : u \in V \wedge P \cdot u : F \cdot u) \quad ?$$

The answer is: preferably with as little additional effort as possible! For this purpose it would be nice if we could *trade* the conjunct $P \cdot u$ to the term $F \cdot u$ of the quantification, because then the above rule (0) and the above calculation remain valid without further ado. (It is possible to formulate and justify a new version of (0), and it is possible to redo that calculation by means of just predicate calculus, but that is too much "ado".) For universal and existential quantification we have such trading rules, but for quantifications in general we don't: this is precluded as, in general, range and term have different types.

For this purpose, we define a new function G, as follows, where $\mathbf{0}$ is $\oplus$'s identity element, for all u:

$$G{\cdot}u \;=\; \begin{array}{ll} \textsf{if} & P{\cdot}u \rightarrow F{\cdot}u \\ [\,] & \neg P{\cdot}u \rightarrow \mathbf{0} \\ \textsf{fi} & \end{array}$$

Function G depends point-wise on predicate P and function F. Therefore, we introduce a binary operator $\mapsto$, mapping a boolean and a function value to a function value, as follows, for all (function values) x:

$$\textsf{true} \mapsto x \;=\; x$$
$$\textsf{false} \mapsto x \;=\; \mathbf{0}$$

Notice that here the value $\mathbf{0}$ is the identity element of $\oplus$, so the meaning of $\mapsto$ implicitly depends on $\oplus$.

Using $\mapsto$ we can rewrite the above definition of G as follows, for all u:

$$G{\cdot}u \;=\; P{\cdot}u \mapsto F{\cdot}u \quad .$$

Now we calculate:

$$\begin{array}{ll} & (\mathbf{Q}u : u{\in}V : P{\cdot}u \mapsto F{\cdot}u) \\ = & \quad \{ \text{ range split } \} \\ & (\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : P{\cdot}u \mapsto F{\cdot}u) \oplus (\mathbf{Q}u : u{\in}V \wedge \neg P{\cdot}u : P{\cdot}u \mapsto F{\cdot}u) \\ = & \quad \{ \text{ definition of } \mapsto \} \\ & (\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \oplus (\mathbf{Q}u : u{\in}V \wedge \neg P{\cdot}u : \mathbf{0}) \\ = & \quad \{ \; \mathbf{0} \text{ is the identity of } \oplus \} \\ & (\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \quad . \end{array}$$

Thus we obtain the following rule for range/term trading, which is applicable independently of the types of $\oplus$ and of the term.

General-Purpose Trading Rule:

$$(\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \;=\; (\mathbf{Q}u : u{\in}V : P{\cdot}u \mapsto F{\cdot}u) \quad .$$

□

2 Simple applications

Universal quantification is based on binary operator $\wedge$, which has true as its identity element. So, in this case the definition of operator $\mapsto$ becomes:

$$\mathsf{true} \mapsto x \;=\; x$$
$$\mathsf{false} \mapsto x \;=\; \mathsf{true}$$

This is equivalent to the definition of (propositional) implication, that is, here $\mapsto$ is just $\Rightarrow$. So, it comes as no surprise now that the rule for range/term trading for universal quantification assumes this form:

$$(\forall u : u{\in}V \wedge P{\cdot}u : Q{\cdot}u) \;=\; (\forall u : u{\in}V : P{\cdot}u \Rightarrow Q{\cdot}u) \quad .$$

Existential quantification is based on binary operator $\vee$, which has false as its identity element. So, in this case the definition of operator $\mapsto$ becomes:

$$\mathsf{true} \mapsto x \;=\; x$$
$$\mathsf{false} \mapsto x \;=\; \mathsf{false}$$

This is equivalent to the definition of conjunction, that is, here $\mapsto$ is just $\wedge$. So, it comes as no surprise now that the rule for range/term trading for existential quantification assumes this form:

$$(\exists u : u{\in}V \wedge P{\cdot}u : Q{\cdot}u) \;=\; (\exists u : u{\in}V : P{\cdot}u \wedge Q{\cdot}u) \quad .$$

This shows why the two well-known rules for range/term trading for universal and existential quantification look so different: because true is the identity element of $\wedge$ and, hence, in this case $y \mapsto x \;=\; y \Rightarrow x$, whereas false is the identity element of $\vee$ and, hence, in this case $y \mapsto x \;=\; y \wedge x$.

* * *

Now we are also able to generalize, with little effort, rule (0) for dummy transformation to the case where the range of quantification contains an additional conjunct:

$$
\begin{array}{ll}
 & (\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \\
= & \quad \{ \text{ General-Purpose Trading Rule } \} \\
 & (\mathbf{Q}u : u{\in}V : P{\cdot}u \mapsto F{\cdot}u) \\
= & \quad \{ \text{ dummy transformation (0) } \} \\
 & (\mathbf{Q}x : x{\in}X : P{\cdot}(\varphi{\cdot}x) \mapsto F{\cdot}(\varphi{\cdot}x)) \\
= & \quad \{ \text{ General-Purpose Trading Rule } \} \\
 & (\mathbf{Q}x : x{\in}X \wedge P{\cdot}(\varphi{\cdot}x) : F{\cdot}(\varphi{\cdot}x)) \quad .
\end{array}
$$

That is, we thus obtain a generalized (and well-known) rule for dummy transformation and, indeed, with little additional effort. The rule states that dummy transformation remains possible, provided the substitution $u := \varphi{\cdot}x$ is also applied to the additional conjunct:

(2) $$(\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \;=\; (\mathbf{Q}x : x{\in}X \wedge P{\cdot}(\varphi{\cdot}x) : F{\cdot}(\varphi{\cdot}x)) \quad .$$

In very much the same way, and requiring only three steps, a rule for range splits can be derived, yielding:

(3) $$\begin{array}{l}(\mathbf{Q}u : u{\in}V{\cup}W \wedge P{\cdot}u : F{\cdot}u) \;= \\ (\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \oplus (\mathbf{Q}u : u{\in}W \wedge P{\cdot}u : F{\cdot}u)\end{array}$$

The derivation in Section 0 can now be generalized as follows:

$$
\begin{array}{ll}
 & (\mathbf{Q}u : u{\in}V{\cup}W \wedge P{\cdot}u : F{\cdot}u) \\
= & \quad \{ \text{ range split, cf. (3) } \} \\
 & (\mathbf{Q}u : u{\in}V \wedge P{\cdot}u : F{\cdot}u) \oplus (\mathbf{Q}u : u{\in}W \wedge P{\cdot}u : F{\cdot}u) \\
= & \quad \{ \text{ dummy transformation, cf. (2) (twice) } \} \\
 & (\mathbf{Q}x : x{\in}X \wedge P{\cdot}(\varphi{\cdot}x) : F{\cdot}(\varphi{\cdot}x)) \oplus (\mathbf{Q}y : y{\in}Y \wedge P{\cdot}(\psi{\cdot}y) : F{\cdot}(\psi{\cdot}y))
\end{array}
$$

3 A more realistic application

In rh274 I formulated the following property −formula (4), on page 1−, but rewritten here with the above symbols for the binary operator and the quantor:

(4) $$(\mathbf{Q}\,t : t{\subseteq}(b{\triangleright}s) : F{\cdot}t) \;=\; (\mathbf{Q}\,t : t{\subseteq}s : F{\cdot}t) \oplus (\mathbf{Q}\,t : t{\subseteq}s : F{\cdot}(b{\triangleright}t)) \quad .$$

I added the remark that this property would remain valid in the presence of an additional conjunct in the ranges of the quantifications. Proving this is a walk-over now, with our new operator $\mapsto$:

$$
\begin{array}{ll}
 & (\,\mathbf{Q}\,t : t\subseteq(b\rhd s)\wedge P\cdot t : F\cdot t\,) \\
= & \quad \{\ \text{General-Purpose Trading Rule}\ \} \\
 & (\,\mathbf{Q}\,t : t\subseteq(b\rhd s) : P\cdot t \mapsto F\cdot t\,) \\
= & \quad \{\ \text{property }(4)\text{, with } P\cdot t\mapsto F\cdot t \text{ for } F\cdot t\ \} \\
 & (\,\mathbf{Q}\,t : t\subseteq s : P\cdot t \mapsto F\cdot t\,) \oplus (\,\mathbf{Q}\,t : t\subseteq s : P\cdot(b\rhd t) \mapsto F\cdot(b\rhd t)\,) \\
= & \quad \{\ \text{General-Purpose Trading Rule (twice)}\ \} \\
 & (\,\mathbf{Q}\,t : t\subseteq s\wedge P\cdot t : F\cdot t\,) \oplus (\,\mathbf{Q}\,t : t\subseteq s\wedge P(b\rhd t) : F\cdot(b\rhd t)\,) \quad .
\end{array}
$$

The appearance of $P\cdot(b\rhd t)$ in the right-hand quantification comes as no surprise, as this is a direct consequence of the presence of $F\cdot(b\rhd t)$ in the right-hand quantification in property (4): there, the subexpression $b\rhd t$ emerges, in the proof of this property, as the result of a dummy transformation. As we have seen earlier, such dummy transformations also affect additional conjuncts in the ranges of quantifications.

Eindhoven, 23 april 2003

Rob R. Hoogerwoord
department of mathematics and computing science
Eindhoven University of Technology
postbus 513
5600 MB Eindhoven