Dummy transformations and the like

The writing of this note was triggered by Wim Feijen who posed me a question about (the shape of) rules for dummy transformation. Before I can discuss these, however, I need a few concepts from relational calculus that enable me to formulate these rules in a relational way.

For a relation R we have:

(0a) "R is total" $\equiv$ $[\text{true} \circ R]$, or (equivalently):
(0b) "R is total" $\equiv$ $[I \Rightarrow \sim R \circ R]$
(1) "R is functional" $\equiv$ $[R \circ \sim R \Rightarrow I]$

I shall not prove here that (0a) and (0b) are indeed equivalent: I have done a point-wise verification and now I do not want to dig into the relational calculus for a point-free proof; I guess it is simple.

A relation R is a <u>function</u> if it is total and functional. Finally, we have:

(2) "R is surjective" $\equiv$ "$\sim R$ is total"

We now have the following lemma. One half of it ("$\Rightarrow$") I have proved 4 years ago in rh146: "Knaster-Tarski in disguise", but that proof was not point-free and disguised (somewhat) the fact that it is essential that R is a function. The other half of it ("$\Leftarrow$") is completely new to me, but I can imagine that it is well-known to the relationalists.

<u>lemma</u> : "R is a function"

$\equiv$

$(\forall X,Y :: [Y \Rightarrow X \circ R] \equiv [Y \circ \sim R \Rightarrow X])$

<u>proof</u>:

"$\Rightarrow$" : by mutual implication:

$[Y \Rightarrow X \circ R]$

$\Rightarrow$ { $\circ$ is monotonic }

$[Y \circ \sim R \Rightarrow X \circ R \circ \sim R]$

$\Rightarrow$ { R is functional : (1) }

$[Y \circ \sim R \Rightarrow X]$

$\Rightarrow$ { $\circ$ is monotonic }

$[Y \circ \sim R \circ R \Rightarrow X \circ R]$

$\Rightarrow$ { R is total : (0b) }

$[Y \Rightarrow X \circ R]$.

"$\Leftarrow$" : this gives 2 proof obligations:

"R is total"

$\equiv$ { (0b) }

$[I \Rightarrow \sim R \circ R]$

$\equiv$ { antecedent: $X,Y := \sim R, I$ }

$[\sim R \Rightarrow \sim R]$

$\equiv$ { }

true ,

and:

"R is functional"

$\equiv$ { (1) }

$$
\begin{array}{ll}
 & [R \circ \sim R \Rightarrow I] \\
\equiv & \quad \{ \text{ antecedent : } X,Y := I,R \} \\
 & [R \Rightarrow R] \\
\equiv & \quad \{ \} \\
 & true \quad .
\end{array}
$$

$\square$

In rh146 I wrote $R@Y$ instead of $Y \circ \sim R$: if R is a function and Y represents a set then $R@Y$ represents the set ("R map Y") of all function values obtained by application of R to an element of Y. By means of the point-wise interpretation I discovered yesterday, and this is new to me too, that $@$ can be defined by:

(3) $\quad [R@Y \equiv Y \circ \sim R]$

* * *

In my Ph.D thesis I formulated the following rule for <u>dummy transformations</u>, here specialised for universal quantifications:

for surjective function F and predicate X :

$$(\forall x :: X.x) \equiv (\forall y :: X.(F.y))$$

In point-free notation this can be formulated as:

$$[X] \equiv [X \circ F]$$

This is, however, nothing but an instance of the above lemma:

$$
\begin{array}{ll}
 & [\; X \circ F \;] \\
\equiv & \quad \{ \text{ lemma: } Y, R := \mathit{true}, F \;\} \\
 & [\; \mathit{true} \circ \sim F \;\Rightarrow\; X \;] \\
\equiv & \quad \{ \text{ } F \text{ is surjective: (2) and (0a) } \} \\
 & [\; X \;] \;.
\end{array}
$$

Wim Feijen told me another rule for dummy transformations, which he attributed to Eric Hehner and which has the charm that the function need not be surjective:

for function F and predicates X, Y:

$$(\forall x : (F@Y).x : X.x) \;\equiv\; (\forall y : Y.y : X.(F.y)) \;,$$

which in point-free notation amounts to:

$$[\; F@Y \Rightarrow X \;] \;\equiv\; [\; Y \Rightarrow X \circ F \;] \;.$$

This too is an instance of the lemma, because by (3) — the definition of @ — we can replace $F@Y$ by $Y \circ \sim F$.

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<u>afterthought</u>: formula (3) (is nice,) because it establishes, via the Galois connection in the lemma, that for functions F $(F@)$ is universally disjunctive (, which is a well-known property of @).

□

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