Why $\sqrt{p}$ is irrational, for every prime p

In this little essay all variables denote positive natural numbers (i.e.: natural numbers at least 1). For fixed prime number p I wish you to take for granted the existence of a function f satisfying

$$\begin{array}{lll} \text{(i)} & f.p & = 1 \\ \text{(ii)} & f.(x*y) & = f.x + f.y \quad , \text{ for all } x, y \; , \end{array}$$

the idea being that f.x is «the number of prime factors p occurring in x» ; a formalised definition of this can be used to prove (i) and (ii) —this probably requires mathematical induction— , but I am perfectly happy with (i) and (ii) as a definition (and the missing $f.q = 0$ for every <u>other</u> prime q , and $f.1 = 0$, but these I don't need).

Now we calculate:

$$\begin{array}{ll} & p * x^2 = y^2 \\ \Rightarrow & \quad \{ \text{ Leibniz } \} \\ & f.(p*x^2) = f.(y^2) \\ \equiv & \quad \{ \text{ (ii) (thrice) and (i) } \} \\ & 1 + 2*f.x = 2*f.y \\ \equiv & \quad \{ \text{ odd numbers differ from even numbers } \} \\ & \text{false} \; , \end{array}$$

from which we conclude that the equation

$$x,y : \; p*x^2 = y^2$$

has no solutions in the positive natural numbers. Hence, no positive naturals x, y exist such that $\sqrt{p} = x/y$.

The above formalisation of f —i.e.: (i) and (ii) and the two missing cases— is not at all far-fetched: every positive natural corresponds in a unique way to a bag of primes —its factorisation—, and the above just defines f recursively over that bag.

Eindhoven, 23 march 1994
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