Yet another solution

From the first sentence of EWD1170 I quote:
"Does there exist an equilateral triangle whose vertices have integer (orthogonal Cartesian) coordinates?".
Not having read the remainder of EWD1170 nor WF179, I solved this problem in 5 minutes this morning, during my bicycle ride to the university and, obviously, without pencil and paper. My solution differs from Edsger's and Wim's.

My first observation was that the middle point of a line segment whose endpoints are grid points is a grid point too, well almost: it is a grid point if the endpoints of the segment have even coordinates; the latter assumption may be safely made without loss of generality, because we can replace the grid by one with twice the resolution of the old grid.
By using the middle point of one of its edges, an equilateral triangle can be (de)composed from two right-angled triangles with angles of 30 and 60 degrees. On account of the previous paragraph, the problem at hand is therefore equivalent to:
"Does there exist a right-angled triangle with angles of 30 and 60 degrees and whose vertices have integer coordinates?".
One may well wonder what the point is of this transformation, but my idea was that reasoning

about a right-angled triangle might be easier in view of the rectangular grid. For example, the grid is invariant under rotations over 90 degrees, provided (of course) the center of rotation is a gridpoint. As a result, <u>every</u> (nondegenerate) right-angled triangle whose vertices are grid points has the following

<u>property</u>: the ratio of the lengths of the rectangular edges of the triangle is a rational number.

<u>proof</u>: by means of a "suitable" rotation over 90 degrees the two rectangular edges can be considered as collinear, and the property holds for any two collinear (nondegenerate) line segments whose endpoints are grid points.

□

This is relevant because a right-angled triangle with angles of 30 and 60 degrees is a right-angled triangle in which the ratio of the (longer and shorter) lengths of its rectangular edges equals $\sqrt{3}$. (Thus we eliminate the awkward notion of 30 and 60 degrees angles.) Now we are done, because $\sqrt{3}$ is not a rational number. Hence, the answer to the problems is: "No, except for the degenerate case (of coinciding vertices)."

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