The function "reverse"

In connection with the work of one of his students, who spent 7 pages on a derivation of what is essentially a definition of the function "reverse" on finite lists, R.C. Backhouse asked how I would perform such a derivation. Here is my attempt.

First, we observe that the type of the elements of the lists is of no relevance to this discussion, so we shall not mention the element type any further and we shall just speak of lists. Here, all lists are finite.

Second, we observe that the construction of a derivation presupposes the availability of a specification, otherwise there is nothing to be derived. In our case, we are interested in a function rev of type $\mathcal{L}_* \rightarrow \mathcal{L}_*$, where $\mathcal{L}_*$ denotes the set of lists. One aspect of rev is that it establishes an isomorphism between $\mathcal{L}_*$ and $\mathcal{L}_*$, but as a specification this is not enough: every permutation establishes such an isomorphism.

Actually, we can make a stronger statement. The set $\mathcal{L}_*$ is the disjoint union of the sets $\mathcal{L}_j$, $0 \leq j$, where $\mathcal{L}_j$ denotes the set of lists of length j. Then we have that rev establishes an isomorphism between $\mathcal{L}_j$ and $\mathcal{L}_j$, for all natural j, and so does every permutation. The specification of rev can now be formulated as follows. Here and in the sequel, b and c denote elements and x denotes a list.

<u>specification of rev</u>: for all natural j,

rev has type $\mathcal{L}_j \to \mathcal{L}_j$, and

$$(\forall x : x \in \mathcal{L}_{j+1} : (\forall i : 0 \leq i \leq j : rev.x.i = x.(j-i)))$$

□

Notice that because $\mathcal{L}_0$ is de singleton set { [] } we need not further specify rev on $\mathcal{L}_0$: from its type $\mathcal{L}_0 \to \mathcal{L}_0$ it follows immediately that rev.[] = [] .

Those who are allergic to indexitis may find this specification ugly ; yet, it allows for a short and easy derivation of a definition for rev.
Moreover, this specification shows immediately that rev is its own inverse, as follows. When we consider lists of length $j+1$ as functions on the interval $[0,j]$ we have that $rev.x = x \circ (j-)$ (for $x \in \mathcal{L}_{j+1}$) and hence:

$$\begin{array}{ll} & rev.(rev.x) \\ = & \{ \text{ the above (twice) } \} \\ & x \circ (j-) \circ (j-) \\ = & \{ \text{ on } [0,j] \text{ we have } (j-) = (j-)^{-1} \} \\ & x \ . \end{array}$$

So, without further proof obligations we conclude that $rev = rev^{-1}$; hence, rev establishes an isomorphism between $\mathcal{L}_{j+1}$ and $\mathcal{L}_{j+1}$.

We already know that rev.[] = [] and for $x \in \mathcal{L}_j$, so $b;x \in \mathcal{L}_{j+1}$, we derive:

$$\begin{array}{ll} & (b;x).(j-j) \\ = & \{ j-j = 0 \ ; \text{ definition of } ; \ \} \end{array}$$

b ,

and, for i : $0 \le i < j$:

$$
\begin{array}{ll}
 & (b;x)\cdot(j-i) \\
= & \quad \{\ \} \\
 & (b;x)\cdot(j-1-i+1) \\
= & \quad \{\ i<j \text{ , so } 0 \le j-1-i \text{ : definition of } ;\ \} \\
 & x\cdot(j-1-i) \\
= & \quad \{\ x \in \mathcal{L}_j \wedge 0 \le i \le j-1 \text{ : spec. of rev } \} \\
 & rev\cdot x\cdot i \quad .
\end{array}
$$

This derivation shows that $rev\cdot(b;x)$ can be defined in terms of b and $rev\cdot x$; that is, rev is a catamorphism. This means that function f exists such that rev can be defined as follows:

$$
\begin{array}{lcl}
rev\cdot[] & = & [] \\
rev\cdot(b;x) & = & f\cdot b\cdot(rev\cdot x) \quad .
\end{array}
$$

Thus defined rev satisfies its specification provided, of course, that we choose the right f ; from the above derivation we obtain the following specification.

<u>specification of f</u>: for all natural j and element b,

$f\cdot b$ has type $\mathcal{L}_j \to \mathcal{L}_{j+1}$, and

$$
\begin{array}{l}
(\forall x : x \in \mathcal{L}_j :\ f\cdot b\cdot x\cdot j = b \quad \wedge \\
\qquad\qquad\qquad (\forall i : 0 \le i < j : f\cdot b\cdot x\cdot i = x\cdot i\,) \\
)
\end{array}
$$

□

Finally, we derive a definition for f. From its required type we know $f.b.[] \in \mathcal{L}_1$ and:

$$\begin{array}{ll} & f.b.[].0 \\ = & \{ \text{ specification of } f \text{, } [] \in \mathcal{L}_0 \ \} \\ & b \\ = & \{ \text{ definition of } ; \ \ ; \ b;[] \in \mathcal{L}_1 \text{ as required } \} \\ & (b;[]).0 \ , \end{array}$$

so we choose $f.b.[] = b;[]$. Furthermore, for $x \in \mathcal{L}_j$ we derive:

$$\begin{array}{ll} & f.b.(c;x).(j+1) \\ = & \{ \text{ specification of } f \text{, } c;x \in \mathcal{L}_{j+1} \ \} \\ & b \\ = & \{ \text{ induction hypothesis , } x \in \mathcal{L}_j \ \} \\ & f.b.x.j \\ = & \{ \text{ definition of } ; \ \} \\ & (\,?\ ;\ f.b.x).(j+1) \ , \end{array}$$

and:

$$\begin{array}{ll} & f.b.(c;x).0 \\ = & \{ \text{ specification of } f \text{, } 0 < j+1 \ \} \\ & (c;x).0 \\ = & \{ \text{ definition of } ; \ \} \\ & c \\ = & \{ \text{ definition of } ; \ \} \\ & (c\,;\ ?\,).0 \ , \end{array}$$

and, for $i : 0 \le i < j$:

$$\begin{array}{ll} & f.b.(c;x).(i+1) \\ = & \{ \text{ specification of } f \text{ , } i+1 < j+1 \ \} \end{array}$$

$$
\begin{array}{ll}
 & (c;x)\cdot(i+1) \\
= & \quad \{ \text{ definition of } ; \ \} \\
 & x\cdot i \\
= & \quad \{ \text{ induction hypothesis, } i<j \ \} \\
 & f\cdot b\cdot x\cdot i \\
= & \quad \{ \text{ definition of } ; \ \} \\
 & (? \ ; f\cdot b\cdot x)\cdot(i+1) \quad .
\end{array}
$$

So, by means of 4 calculations each requiring at most 4 elementary steps we obtain the following definition for f:

$$
\begin{array}{lcl}
f\cdot b\cdot [] & = & b \ ; [] \\
f\cdot b\cdot (c;x) & = & c \ ; f\cdot b\cdot x \quad .
\end{array}
$$

It may very well be that the above derivations may be shortened in an appropriate calculus for catamorphisms, but this is how I do it now. The above clearly shows that the derivation of a definition for rev can be performed entirely by elementary means, and the result is well-known. (As a matter of fact, f implements the "snoc"-operation on "cons-lists".) That rev establishes an isomorphism is true, but this fact follows from rev's specification and it has not played a role in the derivations.

Eindhoven, 27 august 1993
Rob R. Hoogerwoord