<u>e</u>

In this note we derive a program for the computation of e^x, for nonnegative real x; we shall do so by making use of no other knowledge about e than that it satisfies:

(0a) $\quad 1+x \leq e^x \qquad$, for all $x : 0 \leq x$
(0b) $\quad e^x \leq 1+x+x^2 \qquad$, for all $x : 0 \leq x \leq 1$

Upon completion of our design, we shall also have proved that (0a) and (0b) completely <u>define</u> e. Throughout this note, all variables denote nonnegative reals, unless stated otherwise.

In a finite amount of time e^x cannot be computed exactly; it can only be approximated. Therefore, we state our programming problem as follows. For given x and ε, satisfying

$$0 < \varepsilon \leq 1 \quad ,$$

we are interested in a value y satisfying:

(1a) $\quad y \leq e^x$
(1b) $\quad e^x \leq y + e^x * \varepsilon$

<u>remark</u>: The restriction $\varepsilon \leq 1$ is not essential but it saves some case analysis in the following derivation; ε specificies y's <u>relative</u> accuracy.
□

To exploit (0b) we consider the case $x \leq 1$ and we investigate the proposal $y = 1+x$, which by (0a) obviously satisfies (1a). To prove (1b) we derive:

$$
\begin{array}{ll}
 & e^x \leq y + e^x * \varepsilon \\
\equiv & \quad \{ y = 1+x \} \\
 & e^x \leq 1+x+e^x * \varepsilon \\
\Leftarrow & \quad \{ \text{(0b) (using } x \leq 1\text{) and transitivity of } \leq \} \\
 & 1+x+x^2 \leq 1+x+e^x * \varepsilon \\
\equiv & \quad \{ \text{algebra} \} \\
 & x^2 \leq e^x * \varepsilon \\
\Leftarrow & \quad \{ \text{(0a)} \Rightarrow 1 \leq e^x \text{, for simplicity's sake} \} \\
 & x^2 \leq \varepsilon \quad .
\end{array}
$$

The second step in this calculation is the only place where we use (0b). It is tempting to replace (0b) by the simpler $e^x \leq 1+2*x$, for $x : 0 \leq x \leq 1$. This gives rise to the stronger $x \leq \varepsilon$ instead of $x^2 \leq \varepsilon$, and the resulting program will not terminate. Also, $e^x \leq 1+2*x$ and (0a) do not uniquely define e.

Because $\varepsilon \leq 1$ we have $x^2 \leq \varepsilon \Rightarrow x \leq 1$; thus, we conclude that:

$$x^2 \leq \varepsilon \ \wedge \ y = 1+x \quad \Rightarrow \quad \text{(1a)} \wedge \text{(1b)}$$

For the case $x^2 > \varepsilon$ we observe that $x/2 < x$ and that $e^x = (e^{x/2})^2$; to investigate whether we can exploit this, we assume the existence of a value z satisfying, for some $\delta : 0 < \delta \leq 1$:

(2a) $\quad z \leq e^{x/2}$

(2b) $\quad e^{x/2} \leq z + e^{x/2} * \delta$

We now calculate for which δ the proposal $y = z^2$ satisfies (1a) and (1b), by trying to prove these:

$$
\begin{array}{ll}
 & y \leq e^x \\
\equiv & \quad \{ \; y = z^2 \; \} \\
 & z^2 \leq e^x \\
\equiv & \quad \{ \; e^x = (e^{x/2})^2 \; ; \quad {}^2 \text{ is monotonic } \} \\
 & z \leq e^{x/2} \\
\equiv & \quad \{ \; (2a) \; \} \\
 & \mathit{true} \; ,
\end{array}
$$

and:

$$
\begin{array}{ll}
 & e^x \leq y + e^x * \varepsilon \\
\equiv & \quad \{ \; y = z^2 \; \} \\
 & e^x \leq z^2 + e^x * \varepsilon \\
\Leftarrow & \quad \{ \; (2b) \text{ rewritten as } \cdots \leq z \; ; \text{ monotonicity of } {}^2 \; \} \\
 & e^x \leq (e^{x/2} * (1-\delta))^2 + e^x * \varepsilon \\
\equiv & \quad \{ \; \text{algebra} \; , \; (e^{x/2})^2 = e^x \; , \; (0a) \Rightarrow 1 \leq e^x \; \} \\
 & 1 \leq 1 - 2 * \delta + \delta^2 + \varepsilon \\
\equiv & \quad \{ \; \text{algebra} \; \} \\
 & 2 * \delta \leq \delta^2 + \varepsilon \\
\Leftarrow & \quad \{ \; 0 \leq \delta^2 \; , \text{ for simplicity's sake } \} \\
 & \delta \leq \varepsilon / 2 \; .
\end{array}
$$

If we now use $f.\varepsilon.x$ to denote, for given ε and x, the values y satisfying (1a) and (1b) as chosen above, then we obtain the following recursive definition for f:

$$
\begin{array}{llll}
f.\varepsilon.x \; = & \underline{\text{if}} & x^2 \leq \varepsilon & \rightarrow \; 1 + x \\
 & \square & x^2 > \varepsilon & \rightarrow \; (\, f.(\varepsilon/2).(x/2) \,)^2 \\
 & \underline{\text{fi}} & &
\end{array}
$$

This is a correctly recursive —well founded— definition because, for all natural k:

$$x^2 \leq \varepsilon * 2^{k+1} \quad \Rightarrow \quad (x/2)^2 \leq (\varepsilon/2) * 2^k$$

So, $(\underline{\min}\, k : k \in \mathbb{N} \wedge x^2 \leq \varepsilon * 2^k : k)$ is a good variant function.

To obtain a sequential program we introduce function g as a generalisation of function f, with:

$$g.p.\varepsilon.x \;=\; (f.\varepsilon.x)^p \quad , \text{for } p \text{ a power of } 2$$

Then we have $f.\varepsilon.x = g.1.\varepsilon.x$ and in the usual way we obtain the following definition for g:

$$\begin{array}{lll} g.p.\varepsilon.x \;=\; \underline{\text{if}} & x^2 \leq \varepsilon \;\rightarrow & (1+x)^p \\ \; [] & x^2 > \varepsilon \;\rightarrow & g.(p*2).(\varepsilon/2).(x/2) \\ \; \underline{\text{fi}} & & \end{array}$$

For the computation of $(1+x)^p$ we introduce function h with

$$h.p.x \;=\; (1+x)^p \quad , \text{for } p \text{ a power of } 2$$

(This is, from a numerical point of view, more attractive than the simpler $h.p.y = y^p$.) Because p is a power of 2, the following definition suffices:

$$\begin{array}{lll} h.p.x \;=\; \underline{\text{if}} & p = 1 \;\rightarrow & 1+x \\ \;\;\;\;\;\;\;\;\;\;\;\;\;\; [] & p \neq 1 \;\rightarrow & h.(p/2).(2*x+x^2) \\ \;\;\;\;\;\;\;\;\;\;\;\;\;\; \underline{\text{fi}} & & \end{array}$$

From the (tail recursive) definitions of g and h we obtain the following sequential program for the computation of $f.\mathcal{E}.X$, for $\mathcal{E}: 0 < \mathcal{E} \leq 1$:

$$
\begin{array}{l}
\quad p, \varepsilon, x := 1, \mathcal{E}, X \quad \{\text{invariant: } f.\mathcal{E}.X = g.p.\varepsilon.x\} \\
; \underline{\text{do}}\ x^2 > \varepsilon \rightarrow p, \varepsilon, x := p*2, \varepsilon/2, x/2\ \underline{\text{od}} \\
\{ x^2 \leq \varepsilon, \text{ hence } f.\mathcal{E}.X = h.p.x \ (\text{invariant}) \} \\
; \underline{\text{do}}\ p \neq 1 \rightarrow p, x := p/2, x*(2+x)\ \underline{\text{od}} \\
\{ p = 1, \text{ hence } f.\mathcal{E}.X = 1 + x \}
\end{array}
$$

Actually, this program computes $(1+X/p)^p$ for a sufficiently large value of p; that is, the smaller $\mathcal{E}$ is the larger p will be. This forms a proof of the well-known fact that:

$$e^x = \lim_{p \to \infty} (1 + x/p)^p$$

Notice that (0a) and (0b) are all we have used about e to obtain this result.

Amsterdam, 29 march 1992
Rob R. Hoogerwoord
department of mathematics and computing science
Eindhoven University of Technology
postbus 513
5600 MB Eindhoven

<u>post scriptum</u>: The above program requires approximately $2 * {}^2\log.(X/\mathcal{E})$ multiplications, if we only count the ones in $x*(2+x)$.

□