A little exercise in combinator logic (mainly for the record)

Ω is a set on which we have a binary operator $\cdot$ ("dot"). This operator is left-binding, i.e. $x \cdot y \cdot z$ must be read as $(x \cdot y) \cdot z$. Ω is nonempty, for it contains values I and K that have the following properties:

(0) $\quad (\underline{A} x :: I \cdot x = x)$

(1) $\quad (\underline{A} x, y :: K \cdot x \cdot y = x)$.

By means of (0) and (1) we can only conclude equalities between certain elements of Ω. As a matter of fact, (0) and (1) do not preclude the possibility that Ω is a singleton set, in which case we have $I = K$. Because this case is uninteresting we exclude it by postulating (2), which states that Ω is <u>not</u> a singleton set.

(2) $\quad (\underline{A} x :: (\underline{E} y :: y \neq x))$.

<u>lemma 0</u>: $\quad I \neq K$

<u>proof</u>:

$I \neq K$

$\Leftarrow$ { Leibniz (heading for application of (0) and (1)) }

$(\underline{E} x :: I \cdot x \neq K \cdot x)$

$=$ { (0) }

$(\underline{E} x :: x \neq K \cdot x)$

$\Leftarrow$ { Leibniz (heading for application of (1)) }

$(\underline{E} x :: (\underline{E} y :: x \cdot y \neq K \cdot x \cdot y))$

$=$ { (1) }

$(\underline{E} x :: (\underline{E} y :: x \cdot y \neq x))$

$\Leftarrow$ { instantiation $x \leftarrow I$ (keep it simple) }

$$
\begin{array}{ll}
 & (\underline{E}y :: I\cdot y \neq I) \\
= & \quad \{ (0) \} \\
 & (\underline{E}y :: y \neq I) \\
= & \quad \{ (2),\ \text{with } x \leftarrow I \} \\
 & true
\end{array}
$$

□

<u>lemma 1</u>: $(\underline{A}x :: I \neq K\cdot x)$

<u>proof</u>:

$$
\begin{array}{ll}
 & I \neq K\cdot x \\
\Leftarrow & \quad \{ \text{Leibniz} \} \\
 & (\underline{E}y :: I\cdot y \neq K\cdot x\cdot y) \\
= & \quad \{ (0) \text{ and } (1) \} \\
 & (\underline{E}y :: y \neq x) \\
= & \quad \{ (2) \} \\
 & true
\end{array}
$$

□

<u>lemma 2</u>: $(\underline{A}x,y :: K\cdot x = K\cdot y \ \equiv\ x = y)$

<u>proof</u>:

$$
\begin{array}{ll}
 & K\cdot x = K\cdot y \\
\Rightarrow & \quad \{ \text{Leibniz, pick } z \text{ arbitrarily in } \Omega\ (\Omega \neq \emptyset) \} \\
 & K\cdot x\cdot z = K\cdot y\cdot z \\
= & \quad \{ (1) \} \\
 & x = y \\
\Rightarrow & \quad \{ \text{Leibniz} \} \\
 & K\cdot x = K\cdot y
\end{array}
$$

□

*   *

*

We now prove that Ω is <u>infinite</u>, by constructing an infinite sequence, in Ω, all whose elements are different. Before doing so, however, we analyse a little further what meaningful expressions can be formed in terms of I, K, and $\cdot$; after all, these are the raw materials our infinite sequence will be made of.

In view of (0), there is no point in forming expressions of the form $I\cdot x$. Similarly, in view of (1), there is nothing to be gained from expressions of the form $K\cdot x\cdot y$. What remains are I and K, and expressions of the form $K\cdot x$, such as $K\cdot I$, $K\cdot K$, and $K\cdot(K\cdot I)$. Now it is no surprise — we have only little freedom left — that we define sequence $x_i\ (0\leq i)$ as follows:

$$
\begin{aligned}
x_0 &= I \\
x_{i+1} &= K\cdot x_i \quad , \ 0\leq i \quad .
\end{aligned}
$$

We prove that all elements of x are different by proving $(\underline{A}j : 0\leq j : (\underline{A}i : 0\leq i<j : x_i\neq x_j))$. We do so by mathematical induction on j. The case $0=j$ is trivial because of the empty-range rule. For j, $0\leq j$, we prove $(\underline{A}i : 0\leq i<j+1 : x_i\neq x_{j+1})$ by case analysis — driven by the case analysis in x's definition — :

$$
\begin{aligned}
& x_0 \neq x_{j+1} \\
= \quad & \{ \text{definition of } x \} \\
& I \neq K\cdot x_j \\
= \quad & \{ \text{lemma 1} \} \\
& \text{true} \quad ,
\end{aligned}
$$

and for i, $0\leq i<j$:

$$\begin{array}{ll} & x_{i+1} \neq x_{j+1} \\ = & \quad \{ \text{ definition of } x \text{ } \} \\ & K \cdot x_i \neq K \cdot x_j \\ = & \quad \{ \text{ lemma 2 } \} \\ & x_i \neq x_j \\ = & \quad \{ \text{ induction hypothesis } \} \\ & \text{true} \ . \end{array}$$

So much for an exercise in combinator calculus. It shows that the definition of sequence x is less far-fetched than I thought it was a while ago. Lemmata 1 and 2 have been formulated separately to factor out a few properties that have nothing to do with x being a sequence. It is somewhat amazing how far-reaching the consequences of the conjunction of three, all by itself very simple, postulates can be.

Eindhoven, 18 januari 1990
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