A small relational proof, for the record

Recently, Rob Hoogerwoord mentioned a probably well-known theorem on monotonic functions, because he was interested in a relational proof. This note records one such proof.

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The theorem

In what follows, R^+ denotes the transitive closure of relation R. Furthermore, with

f is $R \to S$-monotonic ("R to S monotonic")

we indicate that function f is monotonic with respect to relation R on its domain and relation S on its range. In this terminology, the theorem to be proved is:

Theorem

f is $R \to S$-monotonic $\Rightarrow$ f is $R^+ \to S^+$-monotonic.

■

Remark The theorem originally mentioned by Rob Hoogerwoord stated that, for transitive relation S, we have:

f is $R \to S$-monotonic $\Rightarrow$ f is $R^+ \to S$-monotonic.

This, however, is an immediate consequence of our theorem, since† "S transitive" $\equiv$ $S^+ = S$.

■ Remark.

Some preliminaries

In fact, now all there is to do is to find a relational —or, rather a point-free— rendering of the notion $P{\rightarrow}Q$-monotonicity and then start calculating. Before doing so, however, let me briefly summarize the elementaries of relational calculation. For the time being, such a summary is needed, because there are a lot of choices to be made, both notationally and conceptually, and because as yet the calculational community has not agreed on a specific choice. The choice made here is certainly open to discussion, but such a discussion is beyond the scope of this little note.

The choices made below are based on our view of relations as predicates.

- $x\,R\,y \quad \equiv \quad x$ is related to y by R
- $y = f.x \quad \equiv \quad y\,f\,x$ (function as relation)
- $[R \Rightarrow S] \quad \equiv \quad \langle\forall x,y : x\,R\,y : x\,S\,y\rangle$
- $[R \equiv S] \quad \equiv \quad \langle\forall x,y :: x\,R\,y \equiv x\,S\,y\rangle$

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†: see later, for details needed for a proof of this.

Relational composition and relational converse are denoted by an infix ";" and a prefix "~", respectively, and given by:

- $x\ R;S\ y \ \equiv\ \langle \exists z :: x R z \wedge z S y \rangle$ $\quad (\forall x,y)$
- $x\ {\sim}R\ y \ \equiv\ y\ R\ x$. $\quad (\forall x,y)$

With these we can derive other useful rules like

- $f.x\ R\ y \ \equiv\ x\ {\sim}f;R\ y$
- $x\ R\ f.y \ \equiv\ x\ R;f\ y$,

and

- R transitive $\ \equiv\ [R;R \Rightarrow R]$
- R (right-) functional $\ \equiv\ [R;{\sim}R \Rightarrow I]$,

  where I is the identity of composition, given by $x\ I\ y \ \equiv\ x=y$ $\quad (\forall x,y)$.

We leave the calculus at this and calculate:

$$
\begin{array}{ll}
 & f \text{ is } P{\rightarrow}Q\text{-monotonic} \\
\equiv & \quad \{\text{definition}\} \\
 & \langle \forall x,y : x P y : f.x\ Q\ f.y \rangle \\
\equiv & \quad \{\text{above}\} \\
 & \langle \forall x,y : x P y : x\ {\sim}f;Q;f\ y \rangle \\
\equiv & \quad \{\text{above}\} \\
 & [P \Rightarrow {\sim}f;Q;f] \ .
\end{array}
$$

With P^+ defined to be the least solution of

$$X : [P \vee X;X \Rightarrow X] \ ,$$

we are now ready to prove the theorem.

A proof

$\quad f$ is $R^+ \to S^+$-monotonic

$\equiv \quad$ {def. derived above}

$\quad [R^+ \Rightarrow \sim f; S^+; f]$

$\Leftarrow \quad$ {extremity of R^+}

$\quad [R \vee (\sim f; S^+; f);(\sim f; S^+; f) \Rightarrow \sim f; S^+; f]$

$\Leftarrow \quad$ { f is functional, so $[f;\sim f \Rightarrow J]$; prop. J }

$\quad [R \vee \sim f; S^+; S^+; f \Rightarrow \sim f; S^+; f]$

$\Leftarrow \quad$ { pred. calc , $[S \Rightarrow S^+]$ }

$\quad [R \Rightarrow \sim f; S; f] \ \wedge \ [S^+; S^+ \Rightarrow S^+]$

$\equiv \quad$ {def. derived above ‖ S^+: solves-part}

$\quad f$ is $R \to S$-monotonic .

Epilogue

As the above proof shows, once the preliminaries are over, the calculation is rather straightforward. Where a traditional proof would use induction, we use the extremity of R^+. A traditional proof would, however, not have to refer explicitly to f's being a function.

Finally I want to mention a remark made by Jaap van der Woude. In the relational rendering of f's $P \rightarrow Q$-monotonicity, I seem to use that function f is total, because nowhere do I add the restriction that the f.x's and f.y's that are written down, have to exist. I think that *an* answer to this remark is, that we just take the domain of function f as our universe of discourse, but this certainly is not an answer in the relational spirit.

Waalre, 11th June 1996
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