Cardinality of sets and their powersets: Look, Ma! No case studies.

K. Rustan M. Leino and Berna L. Massingill
California Institute of Technology

19 October 1994

A well-known theorem in mathematics is that any set has more subsets than it has elements. This property can be stated as

For any set S , there is no onto function from S to $\mathcal{P}.S$,

where $\mathcal{P}.S$ denotes the powerset of S .

Proofs of this property usually contain a case study that discriminates between S being the empty set and S being a nonempty set. Attempting to write a calculational proof of this theorem, we, too, found ourselves doing a case study; this one, however, was that of introducing a conjunct of the form $X \vee \neg X$ (the Law of the Excluded Middle), where X expressed that a particular element was in the set that we call $T0$ (see below).

One of the references on our bookshelves is [1]. On page 93, it shows a proof of this theorem that does the same case study as in our first attempt. This proof lacks the S empty *vs.* nonempty case study, an oversight that makes that proof erroneous.

Being unsatisfied with these inferior proofs, we continued our search for a better one. As a result, we constructed the following proof, in which no case study is present.

For the proof, we define set $T0$ by

$$T0 = \{ x \mid x \in S \wedge x \notin f.x \triangleright x \} \quad .$$

Two properties of $T0$ are

$$T0 \subseteq S \qquad (0)$$

$$\langle \forall x \mid x \in S \triangleright x \in T0 \equiv x \notin f.x \rangle \quad . \qquad (1)$$

We now calculate, for any f ,

$$
\begin{array}{ll}
 & f : S \rightarrow \mathcal{P}.S \text{ is onto} \\
= & \quad \{ \text{ def. of onto, and def. of powersets } \} \\
 & \langle \forall T \mid T \subseteq S \triangleright \langle \exists x \mid x \in S \triangleright f.x = T \rangle \rangle \\
\Rightarrow & \quad \{ \text{ instantiate with } T := T0 \text{, using (0) } \} \\
 & \langle \exists x \mid x \in S \triangleright f.x = T0 \rangle \\
= & \quad \{ \text{ trading } \} \\
 & \langle \exists x \mid x \in S \wedge f.x = T0 \triangleright \mathit{true} \rangle \\
= & \quad \{ \text{ (1), since } x \in S \ \}
\end{array}
$$

$$
\begin{array}{ll}
& \langle \exists x \mid x \in S \wedge f.x = T0 \triangleright x \in T0 \equiv x \notin f.x \rangle \\
= & \quad \{ \text{ using } f.x = T0 \text{ from the range } \} \\
& \langle \exists x \mid x \in S \wedge f.x = T0 \triangleright x \in T0 \equiv x \notin T0 \rangle \\
= & \quad \{ \; X \equiv \neg X \equiv \mathit{false} \text{, with } X := x \in T0 \; \} \\
& \langle \exists x \mid x \in S \wedge f.x = T0 \triangleright \mathit{false} \rangle \\
= & \quad \{ \text{ pred. calc. } \} \\
& \mathit{false} \quad .
\end{array}
$$

And thar it is!

Epilogue

A book that, by providing new and nice calculational proofs of its theorems, distinguishes itself from previous math texts is [0]. After writing our proof, we naturally wanted to see what this book had to say about the above theorem. We found the theorem as Theorem (20.21) on page 465. Our proof, it turns out, is essentially the same as theirs.

A difference is that [0] defines $T0$ by (0) and (1), whereas we state those as properties of $T0$ that follow directly from our definition.

Another difference is that [0], like [1], first shows that there exists a function $f: S \rightarrow \mathcal{P}.S$, *viz.*, $f.x = \{x\}$, that is one-to-one. Both references then proceed to prove that no *one-to-one* function $f: S \rightarrow \mathcal{P}.S$ is onto. However, the one-to-one-ness of f is never used. Our proof does not mention anything about functions being one-to-one, because this is not necessary.

References

[0] David Gries and Fred B. Schneider. *A Logical Approach to Discrete Math.* Springer-Verlag, 1994.

[1] P.R. Halmos. *Naive Set Theory.* Springer-Verlag, 1974.