The proof of a card trick

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Consider the following card trick.

> A deck of 52 cards, half black and half red, is arranged such that black and red cards alternate. The deck is cut into two subdecks, which are then merged to form a new deck. Note that the number of cards in the subdecks was either even for both subdecks or odd for both. If both contained an even number of cards, the top card of the new deck is placed at the bottom. Now, the top two cards of the deck are of different colors. Moreover, after removing the top two cards, the next two cards are also of different colors, and so on.

We prove this property. Rather than thinking of first merging the two subdecks, we consider keeping the subdecks as two separate piles. Then, taking two cards off the top of the merged deck in the original problem is the same as taking two cards from one pile or one card from both piles. Also, moving the top card to the bottom, as is done in the original problem when each subdeck has an even number of cards, is the same as taking one card from the top of either pile and putting it at the bottom of either pile.

Now then, the invariant is:

0. Each pile consists of an alternation of black and red cards, and

1. Whenever both piles are non-empty, the top card of each of the two piles differ in color.

Removing two cards from the top of one pile maintains the invariant, and so does removing one card from the top of each pile. The invariant implies, in both cases, that the removed cards are of different colors.

Finally, we justify that the invariant holds initially. Splitting the given deck, in which cards alternate between black and red, into two parts produces two subdecks, each with alternating card colors. Thus, if each of the two subdecks has an odd number of cards, both parts of the invariant hold. If the number of cards in each is even, moving the top card of either pile to the bottom of either pile maintains (0). Moreover, after such an operation, (1) holds, too.

The problem was communicated to me via Jan van de Snepscheut, who received it from Shankar. The card trick is apparently a well-known one.