The Σ calculus Part 4.

Let us conclude (for now) our exploration of the Σ calculus by introducing one new postulate and one more theorem.

The postulate is called the one-point rule:

$$\langle \Sigma x : x = y : f.x \rangle = f.y$$

The one-point rule is highly useful for converting between quantified and unquantified expressions.

Our final theorem appears again and again when deriving algorithms. It's catchword is 'splitting the range':

$$\langle \Sigma k : 0 \le k < N : f.k \rangle = f.0 + \langle \Sigma k : 1 \le k < N : f.k \rangle$$

Proof

$$\begin{array}{ll}
 & \langle \Sigma k : 0 \le k < N : f.k \rangle \\
= & \quad \{\le \text{ in terms of } <\} \\
 & \langle \Sigma k : (0 < k \quad \vee \quad 0 = k) \quad \wedge \quad k < N : f.k \rangle \\
= & \quad \{\text{distributivity}\} \\
 & \langle \Sigma k : (0 < k \quad \wedge \quad k < N) \qquad \vee \qquad (0 = k \quad \wedge \quad k < N) : f.k \rangle \\
= & \quad \{\text{splitting the range; } 0 < k \quad \wedge \quad 0 = k \qquad \equiv \qquad false\} \\
 & \langle \Sigma k : 0 < k < N : f.k \rangle + \langle \Sigma k : 0 = k \quad \wedge \quad k < N : f.k \rangle \\
= & \quad \{\text{integers; } k < N\text{; one-point rule}\} \\
 & \langle \Sigma k : 1 \le k < N : f.k \rangle + f.0
\end{array}$$

End of Proof.

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