The Σ calculus.

Sums are common enough in programming that it is worth our time to become skilled at manipulating them. Here we guve the repetoire of general formulae that are highly useful when dealing with sums.

Summation is a generalisation of addition. Its format is

$$\langle \Sigma dummies : range : term \rangle$$

Here 'dummy' stands for an unordered list of local variables, whose scope is delineated by the outer angled brackets. In what follows x and y will be used to denote dummies of the understood type.

The range is a boolean expression while the term is type integer. Both range and term may depend on the dummies. Such dependence will be indicated explicitly by using a functional notation. e.g. if a range has the form $r.x.y \quad \vee \quad s.x$, it is a disjunction of $r.x.y$ which may depend on x and y and $s.x$ which may depends on x but not y. Similarly X will stand for an integer that depends on none of the dummies. In the sequel we shall use r and s to denote functions from the integers to the boolean scalars and f and g to denote functions from the integers to the integers.

For the sake of brevity, the range *true* shall be omitted. The following postulate, appealed to by the catchword 'trading', tells us how ranges different from *true* may be eliminated:

(0) $\langle \Sigma x : r.x : f.x \rangle = \langle \Sigma x :: [r.x] * f.x \rangle$

where $[b]$ is Iverson's convention given by

$$[b] = if\ b\ then\ 1\ else\ 0$$

Multiplication distributes over summation i.e. we postulate for any X, f

(1) $X * \langle \Sigma x :: f.x \rangle = \langle \Sigma x :: X * f.x \rangle$

We observe

$X * \langle \Sigma x : r.x : f.x \rangle$

= {trading}

$X * \langle \Sigma x :: [r.x] * f.x \rangle$

= {distributivity}

$\langle \Sigma x :: X * [r.x] * f.x \rangle$

= {trading}

$\langle \Sigma x : r.x : X * f.x \rangle$

Hence we have for any X, r, f

(2) $X * \langle \Sigma x : r.x : f.x \rangle = \langle \Sigma x : r.x : X * f.x \rangle$

To be continued...

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